Lesson 3: Addition and Subtraction Formulas

Student Outcomes

- Students prove the subtraction formula for cosine and use their understanding of the properties of the trigonometric functions to derive the remaining addition and subtraction formulas for the sine, cosine, and tangent functions.
- Students use the addition and subtraction formulas to evaluate trigonometric functions to solve problems.

Lesson Notes

In previous lessons, students have used the unit circle to examine the periodicity and symmetry of the sine, cosine, and tangent functions. They have also explored relationships between trigonometric functions and have derived formulas to evaluate the functions for various values of $\theta$. In Algebra II Module 2 Lesson 17, the teacher led the students through a geometric proof that established the sum formula for sine, and the remaining sum and difference formulas followed. In this lesson, students use analytic methods to prove the subtraction formula for the cosine function and then extend the result to the remaining sum and difference formulas using their understanding of the periodicity and symmetry of the functions. Students apply the formulas to evaluate the trigonometric functions for specific values.

Classwork

Opening (3 minutes)

Remind students that they have used the periodicity and symmetry of trigonometric functions to evaluate the sine, cosine, and tangent functions for specific values of $\theta$. Students should then respond to the following prompts. After sharing their thoughts with a partner, several students could discuss their suggestions.

- In the previous two lessons, we have been discussing the unit circle using a carousel model with a rider rotating counterclockwise on the outer edge of the carousel. Suppose the carousel rotates $\frac{\pi}{4}$ radians from its starting position and then stops. How can we determine the position of the rider when the carousel stops?
  - The position is determined by $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$.

- Instead, suppose that the carousel rotates $\frac{\pi}{3}$ radians from its starting position and then stops. How can we determine the position of the rider?
  - The position is determined by $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$. 
Now, suppose that the carousel rotates \( \frac{7\pi}{12} \) radians from its starting position and then stops. How could we use what we have already discovered about the sine, cosine, and tangent functions to determine the position of the rider when the carousel stops? What additional information would we need to determine the position of the rider? Share your responses with a partner.

- Answers will vary but should address rewriting \( \frac{7\pi}{12} \) as a sum or difference of values of \( \theta \) for which we know the exact sine and cosine values (e.g., \( \frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3} \)). Additional information required might include determining a means to find the sine and cosine of a sum.

**Discussion (5 minutes)**

This discussion addresses the fact that evaluating trigonometric functions for a sum of two values \( \alpha + \beta \) is not equivalent to evaluating the trigonometric functions for each value \( \alpha \) and \( \beta \) and then finding the sum. This discussion also address the utility of having a formula to compute the trigonometric functions of any sum or difference.

- One way to evaluate the position of the rider at \( \theta = \frac{7\pi}{12} \) is to rewrite \( \frac{7\pi}{12} \) as a sum of values for which we can calculate the exact positions, for example, \( \frac{7\pi}{12} = \left( \frac{\pi}{4} + \frac{\pi}{3} \right) \). Perhaps we could find the position of our rider at \( \theta = \frac{7\pi}{12} \) by adding the front/back positions and adding the right/left positions of the rider for \( \theta = \frac{\pi}{4} \) and \( \theta = \frac{\pi}{3} \).

- Determine whether this strategy is valid, and either justify why it is valid or provide a counterexample to demonstrate that it is not valid. Share your thoughts with a partner.

  - Answers will vary, but students should determine that summing the position coordinates for \( \theta = \frac{\pi}{4} \) and \( \theta = \frac{\pi}{3} \) does not produce the position coordinates for \( \theta = \frac{7\pi}{12} \). Counterexamples might include:

    - We know that \( \cos \left( \frac{7\pi}{12} \right) \neq \cos \left( \frac{\pi}{4} \right) + \cos \left( \frac{\pi}{3} \right) \) because \( \cos \left( \frac{\pi}{4} \right) + \cos \left( \frac{\pi}{3} \right) = \frac{\sqrt{2}}{2} + \frac{1}{2} \) but \( \cos \left( \frac{7\pi}{12} \right) < 0 \) because rotation by \( \frac{7\pi}{12} \) produces a terminal ray in Quadrant II where cosine is negative. We know that \( \sin \left( \frac{7\pi}{12} \right) \neq \sin \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{3} \right) \) because \( \sin \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \), which exceeds the maximum possible value of 1 for the sine function.

- Now you may remember that in Algebra II we used geometry to establish the formulas for the sums and differences of sine and cosine. Here we are going to use a different approach to prove them.
Example 1 (7 minutes)

This example provides an analytic way to derive the formula for the cosine of a difference. The students then apply their understanding of the properties of the trigonometric functions to the cosine of a difference formula to derive the remaining addition and subtraction formulas and to solve problems. The example should be completed in a whole-class setting. Alternatively, students could complete it in pairs or small groups, and the results could be discussed in a whole-class setting.

- What do the points \( A \) and \( B \) represent?
  - Point \( A \) represents the position on the unit circle after the initial ray is rotated by the amount \( \alpha \), and point \( B \) represents the position on the unit circle after the initial ray is rotated by the amount \( \beta \).

- How can we find the distance between points \( A \) and \( B \)?
  - We can apply the distance formula.

- Which gives us?
  - \[ AB = \sqrt{(\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) - \sin(\beta))^2} \]

- What is the expanded form of this expression?
  - \[ AB = \sqrt{\cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) - 2\sin(\alpha)\sin(\beta) + \sin^2(\beta)} \]

- How can we simplify the expression under the radical?
  - We can use the Pythagorean identity: \( \sin^2(\alpha) + \cos^2(\alpha) = 1 \) and \( \sin^2(\beta) + \cos^2(\beta) = 1 \). We can also factor \(-2\) from the terms \(-2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta)\), which results in the simplified expression \( AB = \sqrt{2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))} \).

- How does the procedure applied to part (b) compare to that in part (a)?
  - The same procedure is applied but this time to the image points.

- Why can we set \( AB \) equal to \( A'B' \)?
  - \( A'B' \) is the image of \( AB \) rotated by \(-\beta\) about the origin, and distance is preserved under rotation, so the length of the segment is the same before and after rotation.

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**Scaffolding:**
- Provide a visual display of the distance formula.
- Show students the expressions for the lengths of \( \overline{AB} \) and \( \overline{A'B'} \) and ask students what the different parts of the expression represent and where they come from in the diagram.
- Have advanced students derive the formula for \( \cos(\beta - \alpha) \) using the same analytic method and explain why the formula is equivalent to that for \( \cos(\alpha - \beta) \).
Example 1

Consider the figures below. The figure on the right is obtained from the figure on the left by rotating by \(-\beta\) about the origin.

a. Calculate the length of \(AB\) in the figure on the left.

\[
AB = \sqrt{\cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) - 2\sin(\alpha)\sin(\beta) + \sin^2(\beta)}
\]

\[
= \sqrt{\cos^2(\alpha) + \sin^2(\alpha) + \cos^2(\beta) + \sin^2(\beta) - 2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta)}
\]

\[
= \sqrt{2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))}
\]

b. Calculate the length of \(A'B'\) in the figure on the right.

\[
A'B' = \sqrt{\cos^2(\alpha - \beta) - 1 + \sin^2(\alpha - \beta)}
\]

\[
= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)}
\]

\[
= \sqrt{\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) + 1 - 2\cos(\alpha - \beta)}
\]

\[
= \sqrt{2 - 2\cos(\alpha - \beta)}
\]

c. Set \(AB\) and \(A'B'\) equal to each other, and solve the equation for \(\cos(\alpha - \beta)\).

\[
\sqrt{2 - 2\cos(\alpha - \beta)} = \sqrt{\frac{2 - 2\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}{2 - 2\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}
\]

\[
(\sqrt{2 - 2\cos(\alpha - \beta)})^2 = \left(\frac{2 - 2\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}{2 - 2\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}\right)^2
\]

\[
2 - 2\cos(\alpha - \beta) = 2 - 2\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)
\]

\[
-2\cos(\alpha - \beta) = -2\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)
\]

\[
\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)
\]
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- Turn to your neighbor and explain how you know that $\overline{AB}$ and $\overline{A'B'}$ are equal and what the formula you have just derived represents.
  - $\overline{AB}$ and $\overline{A'B'}$ have the same length because they are images of each other under rotation, which preserves length. The formula just derived provides a way to calculate the cosine of a number for which the cosine is not readily known by expressing that number as a difference of two numbers for which the sine and cosine values are readily known.

Exercises 1–2 (7 minutes)

Students should complete the exercises in pairs. After an appropriate time, volunteers could display their solutions, and additional students could provide alternative solutions or offer counterarguments to refute a solution. Students establish the sum formula for sine in the Exit Ticket, but it is summarized here for reference.

### Exercises 1–2

1. Use the fact that $\cos(-\theta) = \cos(\theta)$ to determine a formula for $\cos(\alpha + \beta)$.

   $\cos(\alpha + \beta) = \cos(\alpha - \beta)$
   
   $= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta)$
   
   $= \cos(\alpha)\cos(\beta) + \sin(\alpha)(-\sin(\beta))$
   
   $= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

2. Use the fact that $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$ to determine a formula for $\sin(\alpha - \beta)$.

   $\sin(\alpha - \beta) = \cos\left(\frac{\pi}{2} - (\alpha - \beta)\right)$
   
   $= \cos\left(\frac{\pi}{2} - \alpha + \beta\right)$
   
   $= \cos\left(\frac{\pi}{2} - \alpha\right)\cos(\beta) - \sin\left(\frac{\pi}{2} - \alpha\right)\sin(\beta)$
   
   $= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

### Scaffolding:
- Prompt struggling students for Exercise 1 by rewriting $\cos(\alpha + \beta)$ as $\cos(\alpha - (-\beta))$.
- Have advanced students determine the formulas without cueing.

### Sum and Difference Formulas for the Sine and Cosine Functions

For all real numbers $\alpha$ and $\beta$,

- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
Example 2 (5 minutes)

This example demonstrates how to apply the identity of tangent \( \theta \) as the ratio of sine \( \theta \) to cosine \( \theta \) to derive the addition formula for tangent, which students use to evaluate the tangent function for exact values of \( \theta \). The example should be completed in pairs or small groups and then discussed in a whole-class setting.

- How can we use the identity \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \) to help us find an addition formula for tangent?
  - Answers will vary but might indicate that we could rewrite \( \tan(\alpha + \beta) \) as the quotient of \( \sin(\alpha + \beta) \) and \( \cos(\alpha + \beta) \), and we could apply the addition formulas for sine and cosine to determine a formula.

- If we rewrite \( \tan(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \) and apply the addition formulas for sine and cosine, we find the formula \( \tan(\alpha + \beta) = \frac{\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \). How can we determine what to do next to try to find a more user-friendly expression for a tangent addition formula?
  - Answers will vary but might include dividing by a common term so the expressions are written in terms of \( \tan(\alpha) \) and \( \tan(\beta) \). If students do not suggest this tactic, then suggest it to them.

- What if we divide by \( \cos(\alpha)\cos(\beta) \)?
  - Answers will vary but should address that the cosines in the numerators of each term in the expression will reduce with the cosines in \( \cos(\alpha)\cos(\beta) \) so that each term can be written in terms of the tangent function or as 1.

- Are there any restrictions on the values of \( \alpha \) and \( \beta \) for \( \tan(\alpha + \beta) \)?
  - Yes, \( \alpha \) and \( \beta \) cannot sum to \( \frac{\pi}{2} + \pi n \), where \( n \) is an integer because this would result in \( \tan\left(\frac{\pi}{2} + \pi n\right) \), which we have previously determined to be undefined.

- How can we verify this?
  - Answers will vary. An example of an acceptable response is included. If \( \alpha + \beta = \frac{\pi}{2} \), then \( \beta = \frac{\pi}{2} - \alpha \) and \( \tan(\alpha + \beta) = \frac{\sin(\alpha)\cos\left(\frac{\pi}{2} - \alpha\right) + \sin\left(\frac{\pi}{2} - \alpha\right)\cos(\alpha)}{\cos(\alpha)\cos\left(\frac{\pi}{2} - \alpha\right) - \sin(\alpha)\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{\sin(\alpha)\sin(\alpha) + \cos(\alpha)\cos(\alpha)}{\cos(\alpha)\sin(\alpha) - \sin(\alpha)\cos(\alpha)} = \frac{\sin^2(\alpha) + \cos^2(\alpha)}{0} \), which is undefined.

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**Example 2**

Use the identity \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \) to show that \( \tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \).

\[
\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}.
\]

\[
= \frac{\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \cdot \frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} \cdot \frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.
\]
Exercises 3–5 (10 minutes)

Students should complete the exercises in pairs. After an appropriate time, volunteers could display their solutions. As time permits, students should be encouraged to show different approaches to finding the solutions.

3. Verify the identity \( \tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} \) for all \( \alpha - \beta \neq \frac{\pi}{2} + \pi n \).

\[
\tan(\alpha - \beta) = \tan(\alpha + (-\beta)) \\
= \frac{\tan(\alpha) + \tan(-\beta)}{1 - \tan(\alpha)\tan(-\beta)} \\
= \frac{\tan(\alpha) + (-\tan(\beta))}{1 - \tan(\alpha)(-\tan(\beta))} \\
= \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}
\]

4. Use the addition and subtraction formulas to evaluate the expressions shown.

a. \( \cos\left(\frac{5\pi}{12}\right) \)

\[
\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{2\pi}{3}\right) \\
= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{3}\right) \\
= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) \\
= \frac{-\sqrt{2} + \sqrt{6}}{4}
\]

b. \( \sin\left(\frac{23\pi}{12}\right) \)

\[
\sin\left(\frac{23\pi}{12}\right) = \sin\left(\frac{9\pi}{4} - \frac{\pi}{3}\right) \\
= \sin\left(\frac{9\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{9\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \\
= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) \\
= \frac{-\sqrt{2} - \sqrt{6}}{4}
\]

Scaffolding:
Have advanced students verify using the relationship between the sine and cosine functions that \( \cos\left(-\frac{5\pi}{12}\right) = -\sin\left(\frac{23\pi}{12}\right) \).
c. \( \tan \left( \frac{5\pi}{12} \right) \)

\[
\tan \left( \frac{5\pi}{12} \right) = \tan \left( \frac{\pi}{6} + \frac{\pi}{4} \right) \\
= \frac{\tan \left( \frac{\pi}{6} \right) + \tan \left( \frac{\pi}{4} \right)}{1 - \tan \left( \frac{\pi}{6} \right) \tan \left( \frac{\pi}{4} \right)} \\
= \frac{\sqrt{3} + 1}{1 - \left( \frac{\sqrt{3}}{3} \right)} (1) \\
= \frac{\sqrt{3} + 3}{3 - \sqrt{3}} \\
= \frac{12 + 6\sqrt{3}}{6} \\
= 2 + \sqrt{3}
\]

5. Use the addition and subtraction formulas to verify these identities for all real-number values of \( \theta \).

a. \( \sin(\pi - \theta) = \sin(\theta) \)

\[
\sin(\pi - \theta) = \sin(\pi)\cos(\theta) - \sin(\theta)\cos(\pi) = 0(\cos(\theta)) - \sin(\theta)(-1) = \sin(\theta)
\]

b. \( \cos(\pi + \theta) = -\cos(\theta) \)

\[
\cos(\pi + \theta) = \cos(\pi)\cos(\theta) - \sin(\pi)\sin(\theta) = -1(\cos(\theta)) - 0(\sin(\theta)) = -\cos(\theta)
\]

**Sum and Difference Formulas for the Tangent Function**

For all real numbers \( \alpha \) and \( \beta \) for which the expressions are defined,

\[
\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} \quad \text{and} \quad \tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.
\]

**Closing (3 minutes)**

Have students write a response to the prompts below and share their responses with a partner.

- What are the sum and difference formulas for sine and cosine?
  - \( \cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \)
  - \( \cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \)
  - \( \sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \)
  - \( \sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \)
How can the addition and subtraction formulas be used to find the position of the rider from the beginning of the lesson?

- To find the front/back position of the rider, we need to find \( \sin \left( \frac{7\pi}{12} \right) \), which we could rewrite as the difference of two numbers, for example, \( \sin \left( \frac{3\pi}{4} - \frac{\pi}{6} \right) \), and then apply the formula \( \sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta) \) to find the position.

- To find the right/left position of the rider, we need to find \( \cos \left( \frac{7\pi}{12} \right) \), which we could rewrite as the difference of two numbers, for example, \( \cos \left( \frac{3\pi}{4} - \frac{\pi}{6} \right) \), and then apply the formula \( \cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \) to find the position.

Lesson Summary

The sum and difference formulas for sine, cosine, and tangent are summarized below.

For all real numbers \( \alpha \) and \( \beta \) for which the expressions are defined,

\[
\begin{align*}
\cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
\cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\
\sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
\sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
\tan(\alpha - \beta) &= \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} \\
\tan(\alpha + \beta) &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}
\end{align*}
\]

Exit Ticket (5 minutes)
Lesson 3: Addition and Subtraction Formulas

Exit Ticket

1. Prove that \( \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha) \).

2. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.
   a. \( \sin\left(\frac{\pi}{12}\right) \)
   b. \( \tan\left(\frac{13\pi}{12}\right) \)
Exit Ticket Sample Solutions

1. Prove that \( \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha) \).

\[
\begin{align*}
\sin(\alpha + \beta) &= \sin(\alpha - (-\beta)) \\
&= \sin(\alpha) \cos(-\beta) - \sin(-\beta) \cos(\alpha) \\
&= \sin(\alpha) \cos(\beta) - (-\sin(\beta))(\cos(\alpha)) \\
&= \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha)
\end{align*}
\]

2. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.

a. \( \sin\left(\frac{\pi}{12}\right) \)

\[
\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
= \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{3}\right) \\
= \frac{\sqrt{3}}{2} \left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2} \left(\frac{\sqrt{2}}{2}\right) \\
= \frac{\sqrt{6} - \sqrt{2}}{4}
\]

b. \( \tan\left(\frac{13\pi}{12}\right) \)

\[
\tan\left(\frac{13\pi}{12}\right) = \tan\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \\
= \frac{\tan\left(\frac{3\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{3\pi}{4}\right) \tan\left(\frac{\pi}{3}\right)} \\
= \frac{-1 + \sqrt{3}}{1 - (-1)(\sqrt{3})} \\
= \frac{-1 + \sqrt{3}}{1 + \sqrt{3}} \\
= \frac{-4 + 2\sqrt{3}}{-2} \\
= 2 - \sqrt{3}
\]

Problem Set Sample Solutions

1. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.

a. \( \cos\left(\frac{\pi}{12}\right) \)

\[
\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
= \cos\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{4}\right) \\
= \frac{1}{2} \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} \\
= \frac{\sqrt{2} + \sqrt{6}}{4}
\]
b. \( \sin \left( \frac{\pi}{12} \right) \)

\[
\sin \left( \frac{\pi}{12} \right) = \sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right) \\
= \sin \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{4} \right) - \cos \left( \frac{\pi}{3} \right) \sin \left( \frac{\pi}{4} \right) \\
= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\
= \frac{\sqrt{6} - \sqrt{2}}{4}
\]

c. \( \sin \left( \frac{5\pi}{12} \right) \)

\[
\sin \left( \frac{5\pi}{12} \right) = \sin \left( \frac{\pi}{6} + \frac{\pi}{4} \right) \\
= \sin \left( \frac{\pi}{6} \right) \cos \left( \frac{\pi}{4} \right) + \cos \left( \frac{\pi}{6} \right) \sin \left( \frac{\pi}{4} \right) \\
= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
= \frac{\sqrt{2} + \sqrt{6}}{4}
\]

d. \( \cos \left( - \frac{\pi}{12} \right) \)

\[
\cos \left( - \frac{\pi}{12} \right) = \cos \left( \frac{\pi}{4} - \frac{\pi}{3} \right) \\
= \cos \left( \frac{\pi}{4} \right) \cos \left( \frac{\pi}{3} \right) + \sin \left( \frac{\pi}{4} \right) \sin \left( \frac{\pi}{3} \right) \\
= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\
= \frac{\sqrt{2} + \sqrt{6}}{4}
\]

e. \( \sin \left( \frac{7\pi}{12} \right) \)

\[
\sin \left( \frac{7\pi}{12} \right) = \sin \left( \frac{\pi}{4} + \frac{\pi}{3} \right) \\
= \sin \left( \frac{\pi}{4} \right) \cos \left( \frac{\pi}{3} \right) + \cos \left( \frac{\pi}{4} \right) \sin \left( \frac{\pi}{3} \right) \\
= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\
= \frac{\sqrt{2} + \sqrt{6}}{4}
\]

f. \( \cos \left( - \frac{7\pi}{12} \right) \)

\[
\cos \left( - \frac{7\pi}{12} \right) = \cos \left( - \frac{\pi}{4} - \frac{\pi}{3} \right) \\
= \cos \left( - \frac{\pi}{4} \right) \cos \left( \frac{\pi}{3} \right) + \sin \left( - \frac{\pi}{4} \right) \sin \left( \frac{\pi}{3} \right) \\
= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\
= \frac{\sqrt{2} - \sqrt{6}}{4}
\]
g. \( \sin \left( \frac{13\pi}{12} \right) \)

\[
\sin \left( \frac{13\pi}{12} \right) = \sin \left( \frac{3\pi}{4} + \frac{\pi}{3} \right) \\
= \sin \left( \frac{3\pi}{4} \right) \cos \left( \frac{\pi}{3} \right) + \cos \left( \frac{3\pi}{4} \right) \sin \left( \frac{\pi}{3} \right) \\
= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\
= \frac{\sqrt{2} - \sqrt{6}}{4}
\]

h. \( \cos \left( \frac{13\pi}{12} \right) \)

\[
\cos \left( \frac{13\pi}{12} \right) = \cos \left( \frac{\pi}{4} - \frac{\pi}{3} \right) \\
= \cos \left( \frac{\pi}{4} \right) \cos \left( \frac{\pi}{3} \right) + \sin \left( \frac{\pi}{4} \right) \sin \left( \frac{\pi}{3} \right) \\
= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\
= \frac{\sqrt{2} - \sqrt{6}}{4}
\]

i. \( \sin \left( \frac{\pi}{12} \right) \cos \left( \frac{\pi}{12} \right) + \cos \left( \frac{\pi}{12} \right) \sin \left( \frac{\pi}{12} \right) \)

\[
\sin \left( \frac{\pi}{12} \right) \cos \left( \frac{\pi}{12} \right) + \cos \left( \frac{\pi}{12} \right) \sin \left( \frac{\pi}{12} \right) = \sin \left( \frac{\pi}{6} \right) \\
= \frac{1}{2}
\]

j. \( \sin \left( \frac{5\pi}{12} \right) \cos \left( \frac{\pi}{6} \right) - \cos \left( \frac{5\pi}{12} \right) \sin \left( \frac{\pi}{6} \right) \)

\[
\sin \left( \frac{5\pi}{12} \right) \cos \left( \frac{\pi}{6} \right) - \cos \left( \frac{5\pi}{12} \right) \sin \left( \frac{\pi}{6} \right) = \sin \left( \frac{\pi}{4} \right) \\
= \frac{\sqrt{2}}{2}
\]

k. \( \sin \left( \frac{\pi}{8} \right) \cos \left( \frac{\pi}{8} \right) + \cos \left( \frac{\pi}{8} \right) \sin \left( \frac{\pi}{8} \right) \)

\[
\sin \left( \frac{\pi}{8} \right) \cos \left( \frac{\pi}{8} \right) + \cos \left( \frac{\pi}{8} \right) \sin \left( \frac{\pi}{8} \right) = \sin \left( \frac{\pi}{4} \right) \\
= \frac{\sqrt{2}}{2}
\]

l. \( \cos \left( \frac{\pi}{8} \right) \cos \left( \frac{\pi}{8} \right) - \sin \left( \frac{\pi}{8} \right) \sin \left( \frac{\pi}{8} \right) \)

\[
\cos \left( \frac{\pi}{8} \right) \cos \left( \frac{\pi}{8} \right) - \sin \left( \frac{\pi}{8} \right) \sin \left( \frac{\pi}{8} \right) = \cos \left( \frac{\pi}{4} \right) \\
= \frac{\sqrt{2}}{2}
\]
m. \( \cos \left(\frac{\pi}{3}\right) \cos \left(\frac{\pi}{12}\right) + \sin \left(\frac{\pi}{3}\right) \sin \left(\frac{\pi}{12}\right) \)

\( \cos \left(\frac{\pi}{3}\right) \cos \left(\frac{\pi}{12}\right) + \sin \left(\frac{\pi}{3}\right) \sin \left(\frac{\pi}{12}\right) = \cos \left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \)

n. \( \sin \left(\frac{\pi}{3}\right) \cos \left(\frac{\pi}{12}\right) - \cos \left(\frac{\pi}{3}\right) \sin \left(\frac{\pi}{12}\right) \)

\( \sin \left(\frac{\pi}{3}\right) \cos \left(\frac{\pi}{12}\right) - \cos \left(\frac{\pi}{3}\right) \sin \left(\frac{\pi}{12}\right) = \sin \left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \)

2. Figure 2 is obtained from Figure 1 by rotating the angle by \( \alpha \) about the origin.

Use the method shown in Example 1 to show that \( \cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \).
Lesson 3: Addition and Subtraction Formulas

3. Use the sum formula for sine to show that \( \sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \).

\[
\sin(\alpha - \beta) = \sin(\alpha + (-\beta)) = \sin(\alpha)\cos(-\beta) + \cos(\alpha)\sin(-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)
\]

4. Evaluate \( \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \) to show \( \tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \). Use the resulting formula to show that \( \tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)} \).

\[
\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}
\]

It then follows that

\[
\tan(2\alpha) = \tan(\alpha + \alpha) = \frac{\tan(\alpha) + \tan(\alpha)}{1 - \tan(\alpha)\tan(\alpha)} = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}
\]

5. Show \( \tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} \).

\[
\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)} = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}
\]

6. Find the exact value of the following by using addition and subtraction formulas.
   a. \( \tan\left(\frac{\pi}{12}\right) \)

\[
\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = 2 - \sqrt{3}
\]

   b. \( \tan\left(-\frac{\pi}{12}\right) \)

\[
\tan\left(-\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} = -2 + \sqrt{3}
\]

   c. \( \tan\left(\frac{7\pi}{12}\right) \)

\[
\tan\left(\frac{7\pi}{12}\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = -2 - \sqrt{3}
\]
Lesson 3: Addition and Subtraction Formulas

d. $\tan\left(-\frac{13\pi}{12}\right)$

\[
\tan\left(-\frac{13\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{4\pi}{3}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{4\pi}{3}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{4\pi}{3}\right)} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} = -2 + \sqrt{3}
\]

e. $\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{12}\right)}$

\[
\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \tan\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{3}
\]

f. $\frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{12}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{12}\right)}$

\[
\frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{12}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{3} - \frac{\pi}{12}\right) = \tan\left(\frac{3\pi}{12}\right) = \tan\left(\frac{\pi}{4}\right) = 1
\]

g. $\frac{\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{12}\right)}$

\[
\frac{\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{12} + \frac{\pi}{12}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
\]