Lesson 3: Translating Lines

Student Outcomes

- Students learn that when lines are translated, they are either parallel to the given line or they coincide.
- Students learn that translations map parallel lines to parallel lines.

Classwork

Exercise 1 (3 minutes)

Students complete Exercise 1 independently in preparation for the discussion that follows.

Students should realize that they can only draw one line through point $P$ that is parallel to $L$.

Discussion (3 minutes)

Bring out a fundamental assumption about the plane (as observed in Exercise 1):

- Given a line $L$ and a point $P$ not lying on $L$, there is at most one line passing through $P$ and parallel to $L$.
  - Based on what we have learned up to now, we cannot prove or explain this, so we have to simply agree that this is one of the starting points in the study of the plane.
  - This idea plays a key role in everything we do in the plane. A first consequence is that given a line $L$ and a point $P$ not lying on $L$, we can now refer to the line (because we agree there is only one) passing through $P$ and parallel to $L$. 

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Exercises 2–4 (9 minutes)

Students complete Exercises 2–4 independently in preparation for the discussion that follows.

2. Translate line \( L \) along the vector \( \overrightarrow{AB} \). What do you notice about \( L \) and its image, \( L' \)?

\[ L \] and \( L' \) coincide. \( L = L' \).

3. Line \( L \) is parallel to vector \( \overrightarrow{AB} \). Translate line \( L \) along vector \( \overrightarrow{AB} \). What do you notice about \( L \) and its image, \( L' \)?

\[ L \] and \( L' \) coincide, again. \( L = L' \).

4. Translate line \( L \) along the vector \( \overrightarrow{AB} \). What do you notice about \( L \) and its image, \( L' \)?

\[ L \parallel L' \]

Discussion (15 minutes)

- Now we examine the effect of a translation on a line. Thus, let line \( L \) be given. Again, let the translation be along a given \( \overrightarrow{AB} \), and let \( L' \) denote the image line of the translated \( L \). We want to know what \( L' \) is relative to \( \overrightarrow{AB} \) and line \( L \).

- If \( L = L_{AB} \), or \( L \parallel L_{AB} \), then \( L' = L \).

  - If \( L = L_{AB} \), then this conclusion follows directly from the work in Lesson 2, which says if \( C \) is on \( L_{AB} \), then so is \( C' \); therefore, \( L' = L_{AB} \) and \( L = L' \) (Exercise 2).

  - If \( L \parallel L_{AB} \) and \( C \) is on \( L \), then it follows from the work in Lesson 2, which says that \( C' \) lies on the line \( l \) passing through \( C \) and parallel to \( L_{AB} \). However, \( L \) is given as a line passing through \( C \) and parallel to \( L_{AB} \), so the fundamental assumption that there is just one line passing through a point, parallel to a line (Exercise 1), implies \( l = L \). Therefore, \( C' \) lies on \( L \) after all, and the translation maps every point of \( L \) to a point of \( L' \). Therefore, \( L = L' \) again (Exercise 3).

Scaffolding:
Refer to Exercises 2–4 throughout the discussion and in the summary of findings about translating lines.

Note to Teacher:
The notation \( \text{Translation}(L) \) is used as a precursor to the notation students encounter in high school Geometry (i.e., \( T(L) \)). We want to make clear the basic rigid motion that is being performed, so the notation \( \text{Translation}(L) \) is written to mean the translation of \( L \) along the specified vector.
Caution: One must not over-interpret the equality \( \text{Translation}(L) = L \) (which is the same as \( L = L' \)).

All the equality says is that the two lines \( L \) and \( L' \) coincide completely. It is easy (but wrong) to infer from the equality \( \text{Translation}(L) = L \) that for any point \( P \) on \( L \), \( \text{Translation}(P) = P \). Suppose the vector \( \overrightarrow{AB} \) lying on \( L \) is not the zero vector (i.e., assume \( A \neq B \)). Trace the line \( L \) on a transparency to obtain a red line \( L \), and now slide the transparency along \( \overrightarrow{AB} \). Then, the red line, as a line, coincides with the original \( L \), but clearly every point on \( L \) has been moved by the slide (the translation). Indeed, as we saw in Example 2 of Lesson 2, \( \text{Translation}(A) = B \neq A \). Therefore, the equality \( L' = L \) only says that for any point \( C \) on \( L \), \( \text{Translation}(C) \) is also a point on \( L \), but as long as \( \overrightarrow{AB} \) is not a zero vector, \( \text{Translation}(C) \neq C \).

Strictly speaking, we have not completely proved \( L = L' \) in either case. To explain this, let us define what it means for two geometric figures \( F \) and \( G \) to be equal, that is, \( F = G \): it means each point of \( F \) is also a point of \( G \); conversely, each point of \( G \) is also a point of \( F \). In this light, all we have shown above is that if every point \( C' \) of \( L' \) belongs to \( L \), then \( Q \) is also a point of \( L' \). To show the latter, we have to show that this \( Q \) is equal to \( \text{Translation}(P) \) for some \( P \) on \( L \). This then completes the reasoning.

However, at this point of students’ education in geometry, it may be prudent not to bring up such a sticky point because they are already challenged with all of the new ideas and definitions. Simply allow the preceding reasoning to stand for now, and clarify later in the school year when students are more comfortable with the geometric environment.

- Next, if \( L \) is neither \( L_{AB} \) nor parallel to \( L_{AB} \), then \( L' \parallel L \).
- If we use a transparency to see this translational image of \( L \) by the stated translation, then the pictorial evidence is clear: the line \( L \) moves in a parallel manner along \( \overrightarrow{AB} \), and a typical point \( C \) of \( L \) is translated to a point \( C' \) of \( L' \). The fact that \( L' \parallel L \) is unmistakable, as shown. In the classroom, students should be convinced by the pictorial evidence. If so, leave it at that (Exercise 4).

Here is a simple proof, but to present it in class, begin by asking students how they would prove that two lines are parallel. Ensure students understand that they have no tools in their possession to accomplish this goal. It is only then that they see the need for invoking a proof by contradiction (see discussion above). If there are no obvious ways to do something, then you just have to do the best you can by trying to see what happens if you assume the opposite is true. Thus, if \( L' \) is not parallel to \( L \), then they intersect at a point \( C' \). Since \( C' \) lies on \( L' \), it follows from the definition of \( L' \) (as the image of \( L \) under the translation \( T \)) that there is a point \( C \) on \( L \) so that \( \text{Translation}(C) = C' \).

It follows from Lesson 2 that \( L_{CC'} \parallel L_{AB} \). However, both \( C \) and \( C' \) lie on \( L \), so \( L_{CC'} \parallel L \), and we get \( L \parallel L_{AB} \). This contradicts the assumption that \( L \) is not parallel to \( L_{AB} \), so \( L \) could not possibly intersect \( L' \). Therefore, \( L' \parallel L \).

- Note that a translation maps parallel lines to parallel lines. More precisely, consider a translation \( T \) along a vector \( \overrightarrow{AB} \). Then:
  
  \[ \text{If } L_1 \text{ and } L_2 \text{ are parallel lines, so are } \text{Translation}(L_1) \text{ and } \text{Translation}(L_2). \]
- The reasoning is the same as before: Copy \( L_1 \) and \( L_2 \) onto a transparency, and then translate the transparency along \( AB \). If \( L_1 \) and \( L_2 \) do not intersect, then their red replicas on the transparency will not intersect either, no matter what \( AB \) is used. So, \( \text{Translation}(L_1) \) and \( \text{Translation}(L_2) \) are parallel.

- These findings are summarized as follows:
  - Given a translation \( T \) along a vector \( AB \), let \( L \) be a line, and let \( L' \) denote the image of \( L \) by \( T \).
    - If \( L \parallel L_{\text{AB}} \) or \( L = L_{\text{AB}} \), then \( L' \parallel L \).
    - If \( L \) is neither parallel to \( L_{\text{AB}} \) nor equal to \( L_{\text{AB}} \), then \( L' \parallel L \).

**Exercises 5–6 (5 minutes)**

Students complete Exercises 5 and 6 in pairs or small groups.

5. Line \( L \) has been translated along vector \( AB \), resulting in \( L' \). What do you know about lines \( L \) and \( L' \)?

\[ L \parallel T(L) \]

6. Translate \( L_1 \) and \( L_2 \) along vector \( DE \). Label the images of the lines. If lines \( L_1 \) and \( L_2 \) are parallel, what do you know about their translated images?

Since \( L_1 \parallel L_2 \), then \( (L_1)' \parallel (L_2)' \).
Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We know that there exists just one line, parallel to a given line and through a given point not on the line.
- We know that translations map parallel lines to parallel lines.
- We know that when lines are translated, they are either parallel to the given line or they coincide.

Lesson Summary

- Two lines in the plane are parallel if they do not intersect.
- Translations map parallel lines to parallel lines.
- Given a line $L$ and a point $P$ not lying on $L$, there is at most one line passing through $P$ and parallel to $L$.

Exit Ticket (5 minutes)
Lesson 3: Translating Lines

Exit Ticket

1. Translate point $Z$ along vector $\overrightarrow{AB}$. What do you know about the line containing vector $\overrightarrow{AB}$ and the line formed when you connect $Z$ to its image $Z'$?

2. Using the above diagram, what do you know about the lengths of segments $ZZ'$ and $AB$?

3. Let points $A$ and $B$ be on line $L$ and the vector $\overrightarrow{AC}$ be given, as shown below. Translate line $L$ along vector $\overrightarrow{AC}$. What do you know about line $L$ and its image, $L'$? How many other lines can you draw through point $C$ that have the same relationship as $L$ and $L'$? How do you know?
Exit Ticket Sample Solutions

1. Translate point $Z$ along vector $\vec{AB}$. What do you know about the line containing vector $\vec{AB}$ and the line formed when you connect $Z$ to its image $Z'$?

   The line containing vector $\vec{AB}$ and $ZZ'$ is parallel.

2. Using the above diagram, what do you know about the lengths of segment $ZZ'$ and segment $AB$?

   The lengths are equal: $|ZZ'| = |AB|$.

3. Let points $A$ and $B$ be on line $L$ and the vector $\vec{AC}$ be given, as shown below. Translate line $L$ along vector $\vec{AC}$. What do you know about line $L$ and its image, $L'$? How many other lines can you draw through point $C$ that have the same relationship as $L$ and $L'$? How do you know?

   $L$ and $L'$ are parallel. There is only one line parallel to line $L$ that goes through point $C$. The fact that there is only one line through a point parallel to a given line guarantees it.
Problem Set Sample Solutions

1. Translate ∠XYZ, point A, point B, and rectangle HIJK along vector EF. Sketch the images, and label all points using prime notation.

2. What is the measure of the translated image of ∠XYZ? How do you know?
   
   The measure is 38°. Translations preserve angle measure.

3. Connect B to B'. What do you know about the line that contains the segment formed by BB' and the line containing the vector EF?
   
   BB' \parallel EF.

4. Connect A to A'. What do you know about the line that contains the segment formed by AA' and the line containing the vector EF?
   
   AA' and EF coincide.

5. Given that figure HIJK is a rectangle, what do you know about lines that contain segments HI and JK and their translated images? Explain.
   
   Since HIJK is a rectangle, I know that HI \parallel JK. Since translations map parallel lines to parallel lines, then H'I' \parallel J'K'.

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