Lesson 3: Numbers in Exponential Form Raised to a Power

Student Outcomes

- Students know how to take powers of powers. Students know that when a product is raised to a power, each factor of the product is raised to that power.
- Students write simplified, equivalent numeric, and symbolic expressions using this new knowledge of powers.

Lesson Notes

As with Lesson 2, consider providing opportunities for students to discover the property of exponents introduced in this lesson prior to giving the mathematical rationale as to why it is true. For example, you may present students with the problems in Examples 1 and 2 and allow them to share their thinking about what the answer should be and then provide the mathematical reasoning behind their correct solutions.

We continue the work of knowing and applying the properties of integer exponents to generate equivalent expressions in this lesson. As with Lesson 2, students should be exposed to general arguments as to why the properties are true and be able to explain them on their own with concrete numbers. However, the relationship between the laws of exponents and repeated addition is not as important and could be omitted if time is an issue. The discussion that relates taking a power to a power and the four arithmetic operations may also be omitted, but do allow time for students to consider the relationship demonstrated in the concrete problems \((5 \times 8)^{17}\) and \(5^{17} \times 8^{17}\).

Classwork

Discussion (10 minutes)

Suppose we add 4 copies of 3, thereby getting \((3 + 3 + 3 + 3)\) and then add 5 copies of the sum. We get
\[
(3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3).
\]
Now, by the definition of multiplication, adding 4 copies of 3 is denoted by \((4 \times 3)\),
\[
(4 \times 3) + (4 \times 3) + (4 \times 3) + (4 \times 3),
\]
and also by definition of multiplication, adding 5 copies of this product is then denoted by \(5 \times (4 \times 3)\). So,
\[
5 \times (4 \times 3) = (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3).
\]
A closer examination of the right side of the above equation reveals that we are adding 3 to itself 20 times (i.e., adding 3 to itself \((5 \times 4)\) times). Therefore,
\[
5 \times (4 \times 3) = (5 \times 4) \times 3.
\]

So ultimately, because multiplying can be considered as repeated addition, multiplying three numbers is really repeated addition of a value represented by repeated addition.
Now, let us consider repeated multiplication.

(For example, \((3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3)\) \cdots \(5\) times \(3 \times 3 \times 3 \times 3\) \(5\) times \(= 3^4 \times 3^4 \cdots \times 3^4\).)

- What is multiplying 4 copies of 3 and then multiplying 5 copies of the product?
  - Multiplying 4 copies of 3 is \(3^4\), and multiplying 5 copies of the product is \((3^4)^5\). We wish to say this is equal to \(3^x\) for some positive integer \(x\). By the analogy initiated in Lesson 1, the \(5 \times 4\) in \((5 \times 4) \times 3\) should correspond to the exponent \(x\) in \(3^x\); therefore, the answer should be 
    \[(3^4)^5 = 3^{5 \times 4}.
    \]

This is correct because

\[(3^4)^5 = (3 \times 3 \times 3 \times 3)^5
= (3 \times 3 \times 3 \times 3) \times \cdots \times (3 \times 3 \times 3 \times 3)
= 3 \times 3 \times \cdots \times 3
= 3^{5 \times 4}.
\]

**Examples 1–2**

Work through Examples 1 and 2 in the same manner. (Supplement with additional examples if needed.) Have students calculate the resulting exponent; however, emphasis should be placed on the step leading to the resulting exponent, which is the product of the exponents.

**Example 1**

\[(7^2)^6 = (7 \times 7)^6
\]

\[= (7 \times 7) \times \cdots \times (7 \times 7)
= 7 \times \cdots \times 7
= 7^{6 \times 2}
\]

**Example 2**

\[(1.3^3)^10 = (1.3 \times 1.3 \times 1.3)^{10}
\]

\[= (1.3 \times 1.3 \times 1.3) \times \cdots \times (1.3 \times 1.3 \times 1.3)
= 1.3 \times \cdots \times 1.3
= 1.3^{10 \times 3}
\]

In the same way, we have

For any number \(x\) and any positive integers \(m\) and \(n\),

\[(x^m)^n = x^{mn}\]

because

\[(x^m)^n = (x \times x \cdots x)^n
\]

\[= (x \cdot x \cdots x) \times \cdots \times (x \cdot x \cdots x)
= x^{mn}.
\]
Exercises 1–6 (10 minutes)

Have students complete Exercises 1–4 independently. Check their answers, and then have students complete Exercises 5–6.

Exercise 1
\[(15^3)^9 = 15^{9 \times 3}\]

Exercise 2
\[((-2)^5)^8 = (-2)^{8 \times 5}\]

Exercise 3
\[(3.4^{17})^4 = 3.4^{4 \times 17}\]

Exercise 4
Let \(s\) be a number.
\[(s^{17})^4 = s^{4 \times 17}\]

Exercise 5
Sarah wrote \((3^5)^7 = 3^{12}\). Correct her mistake. Write an exponential expression using a base of 3 and exponents of 5, 7, and 12 that would make her answer correct.

Correct way: \((3^5)^7 = 3^{35}\);
Rewritten Problem: \(3^5 \times 3^7 = 3^{5+7} = 3^{12}\).

Exercise 6
A number \(y\) satisfies \(y^2 - 256 = 0\). What equation does the number \(x = y^4\) satisfy?

Since \(x = y^4\), then \((x)^6 = (y^4)^6\). Therefore, \(x = y^4\) would satisfy the equation \(x^6 - 256 = 0\).

Discussion (10 minutes)

From the point of view of algebra and arithmetic, the most basic question about raising a number to a power has to be the following: How is this operation related to the four arithmetic operations? In other words, for two numbers \(x\), \(y\) and a positive integer \(n\),

1. How is \((xy)^n\) related to \(x^n\) and \(y^n\) ?
2. How is \((\frac{x}{y})^n\) related to \(x^n\) and \(y^n\), \(y \neq 0\) ?
3. How is \((x + y)^n\) related to \(x^n\) and \(y^n\) ?
4. How is \((x - y)^n\) related to \(x^n\) and \(y^n\) ?

The answers to the last two questions turn out to be complicated; students learn about this in high school under the heading of the binomial theorem. However, they should at least be aware that, in general,

\[(x + y)^n \neq x^n + y^n, \text{ unless } n = 1.\]  For example, \((2 + 3)^2 \neq 2^2 + 3^2\).

Allow time for discussion of Problem 1. Students can begin by talking in partners or small groups and then share with the class.
Some students may want to simply multiply $5 \times 8$, but remind them to focus on the above-stated goal, which is to relate $(5 \times 8)^{17}$ to $5^{17}$ and $8^{17}$. Therefore, we want to see 17 copies of 5 and 17 copies of 8 on the right side. Multiplying $5 \times 8$ would take us in a different direction.

$$(5 \times 8)^{17} = (5 \times 8) \times \cdots \times (5 \times 8)$$

17 times

$$= (5 \times \cdots \times 5) \times (8 \times \cdots \times 8)$$

17 times

$$= 5^{17} \times 8^{17}$$

The following computation is a different way of proving the equality.

$$5^{17} \times 8^{17} = (5 \times \cdots \times 5) \times (8 \times \cdots \times 8)$$

17 times

$$= (5 \times 8) \times \cdots \times (5 \times 8)$$

17 times

$$= (5 \times 8)^{17}$$

Answer to Problem 1:

*Because in $(xy)^n$, the factors $x$ and $y$ are repeatedly multiplied $n$ times, resulting in factors of $x^n$ and $y^n$:*

$$(xy)^n = x^n y^n$$

because

$$(xy)^n = (xy) \cdots (xy)$$

$n$ times

$$= (x \cdot x \cdots x) \cdot (y \cdot y \cdots y)$$

$n$ times

$$= x^n y^n$$

Scaffolding:

Advanced learners may ask about cases in which $n$ is not a positive integer. At this point in the module, some students may have begun to develop an intuition about what other integer exponents mean. Encourage them to continue thinking as we begin examining zero exponents in Lesson 4 and negative integer exponents in Lesson 5.
Exercises 7–13 (10 minutes)

Have students complete Exercises 17–12 independently and then check their answers.

<table>
<thead>
<tr>
<th>Exercise 7</th>
<th>Exercise 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>((11 \times 4)^9 = 11^{9\times1} \times 4^{9\times1})</td>
<td>Let (x) be a number. ((5x)^7 = 5^{7\times1} \cdot x^{7\times1})</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Exercise 8</th>
<th>Exercise 11</th>
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</thead>
<tbody>
<tr>
<td>((3^2 \times 7^4)^3 = 3^{2\times3} \times 7^{4\times3})</td>
<td>Let (x) and (y) be numbers. ((5xy^2)^7 = 5^{7\times1} \cdot x^{7\times1} \cdot y^{7\times2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Exercise 9</th>
<th>Exercise 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let (a), (b), and (c) be numbers. ((3^2a^3)^5 = 3^{2\times5} \cdot a^{3\times5})</td>
<td>Let (a), (b), and (c) be numbers. ((5^2bc^3)^4 = 5^{2\times4} \cdot b^{4\times1} \cdot c^{3\times4})</td>
</tr>
</tbody>
</table>

Have students work in pairs or small groups on Exercise 13 after you present the problem.

First ask students to explain why we must assume \(y \neq 0\). They should say that if the denominator were zero then the value of the fraction would be undefined.

- The answer to the fourth question is similar to the third: If \(x\), \(y\) are any two numbers, such that \(y \neq 0\) and \(n\) is a positive integer, then
  \[
  \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.
  \]

Exercise 13

Let \(x\) and \(y\) be numbers, \(y \neq 0\), and let \(n\) be a positive integer. How is \(\left(\frac{x}{y}\right)^n\) related to \(x^n\) and \(y^n\)?

**Because**

\[
\left(\frac{x}{y}\right)^n = \frac{x}{y} \times \cdots \times \frac{x}{y} \quad \text{(n times)}
\]

\[
= \frac{x^n}{y^n} \quad \text{By definition}
\]

**Scaffolding:**

- Have students review problems just completed.
- Remind students to begin with the definition of a number raised to a power. 
Let students know that this type of reasoning is required to prove facts in mathematics. They should always supply a reason for each step or at least know the reason the facts are connected. Further, it is important to keep in mind what we already know in order to figure out what we do not know. Students are required to write two proofs for the Problem Set that are extensions of the proofs they have done in class.

**Closing (2 minutes)**

Summarize, or have students summarize, the lesson. Students should state that they now know how to take powers of powers.

**Exit Ticket (3 minutes)**
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Exit Ticket

Write each expression as a base raised to a power or as the product of bases raised to powers that is equivalent to the given expression.

1. \((9^3)^6 =\)

2. \((113^2 \times 37 \times 51^4)^3 =\)

3. Let \(x, y, z\) be numbers. \((x^2 y z^4)^3 =\)

4. Let \(x, y, z\) be numbers and let \(m, n, p, q\) be positive integers. \((x^m y^n z^p)^q =\)

5. \(\frac{4^8}{5^8} =\)
Exit Ticket Sample Solutions

Write each expression as a base raised to a power or as the product of bases raised to powers that is equivalent to the given expression.

1. \((9^3)^6 = (9^{6\times3}) = 9^{18}\)

2. \((113^2 \times 37 \times 51^4)^3 = (113^2 \times 37 \times 51^4)^3 = (113^2 \times 37 \times 51^4)^3 = (113^2)^3 \times 37^3 \times (51^4)^3 = 113^6 \times 37^3 \times 51^{12}\)  
   - By associative law
   - Because \((xy)^n = x^n y^n\) for all numbers \(x, y\)
   - Because \((xy)^n = x^n y^n\) for all numbers \(x, y\)
   - Because \((x^m)^n = x^{mn}\) for all numbers \(x\)

3. Let \(x, y, z\) be numbers. \((x^2yz^4)^3 = (x^2 \times y \times z^4)^3 = (x^2)^3 \times y^3 \times (z^4)^3 = x^6 \times y^3 \times z^{12}\)  
   - By associative law
   - Because \((xy)^n = x^n y^n\) for all numbers \(x, y\)
   - Because \((xy)^n = x^n y^n\) for all numbers \(x, y\)
   - Because \((x^m)^n = x^{mn}\) for all numbers \(x\)

4. Let \(x, y, z\) be numbers and let \(m, n, p, q\) be positive integers. \((x^m y^n z^p)^q = (x^m \times y^n \times z^p)^q\)  
   - By associative law
   - Because \((xy)^n = x^n y^n\) for all numbers \(x, y\)
   - Because \((xy)^n = x^n y^n\) for all numbers \(x, y\)
   - Because \((x^m)^n = x^{mn}\) for all numbers \(x\)

5. \(\frac{4^8}{5^8} = \left(\frac{4}{5}\right)^8\)
Problem Set Sample Solutions

1. Show (prove) in detail why \((2 \cdot 3 \cdot 7)^4 = 2^4 3^4 7^4\).

\[
(2 \cdot 3 \cdot 7)^4 = (2 \cdot 3 \cdot 7)(2 \cdot 3 \cdot 7)(2 \cdot 3 \cdot 7)(2 \cdot 3 \cdot 7)
\]

By definition

\[
= (2 \cdot 2 \cdot 2)(3 \cdot 3 \cdot 3)(7 \cdot 7 \cdot 7)
\]

By repeated use of the commutative and associative properties

\[
= 2^4 3^4 7^4
\]

By definition

2. Show (prove) in detail why \((xyz)^4 = x^4 y^4 z^4\) for any numbers \(x, y, z\).

The left side of the equation \((xyz)^4\) means \((xyz)(xyz)(xyz)(xyz)\). Using the commutative and associative properties of multiplication, we can write \((xyz)(xyz)(xyz)(xyz)\) as \((xxx)(yyy)(zzz)\), which in turn can be written as \(x^4 y^4 z^4\), which is what the right side of the equation states.

3. Show (prove) in detail why \((xyz)^n = x^n y^n z^n\) for any numbers \(x, y, z\) and for any positive integer \(n\).

Beginning with the left side of the equation, \((xyz)^n\) means \((xyz)(xyz)(xyz)\) \(n\) times. Using the commutative and associative properties of multiplication, \((xyz)(xyz)(xyz)\) can be rewritten as \(x \cdot x \cdot x \cdot \ldots \cdot x \cdot y \cdot y \cdot y \cdot \ldots \cdot y \cdot z \cdot z \cdot z \cdot \ldots \cdot z\) \(n\) times, and, finally, \(x^n y^n z^n\), which is what the right side of the equation states. We can also prove this equality by a different method, as follows. Beginning with the right side \(x^n y^n z^n\) means \((x \cdot x \cdot x \cdot \ldots \cdot x \cdot y \cdot y \cdot y \cdot \ldots \cdot y \cdot z \cdot z \cdot z \cdot \ldots \cdot z)\) \(n\) times, which by the commutative property of multiplication can be rewritten as \((xyz)(xyz)(xyz)\) \(n\) times. Using exponential notation, \((xyz)(xyz)(xyz)\) \(n\) times can be rewritten as \((xyz)^n\), which is what the left side of the equation states.