Lesson 10: The Power of Algebra—Finding Pythagorean Triples

Student Outcomes

- Students explore the difference of two squares identity $x^2 - y^2 = (x - y)(x + y)$ in the context of finding Pythagorean triples.

Lesson Notes

This lesson addresses standards A-SSE.A.2 and A-APR.C.4, and MP.7 directly. In particular, this lesson investigates the example suggested by A-APR.C.4: Show how “the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.” This polynomial identity is proven in this lesson using the difference of two squares identity by

$$(x^2 + y^2)^2 - (x^2 - y^2)^2 = (x^2 + y^2 - x^2 + y^2)(x^2 + y^2 + x^2 - y^2) = (2y^2)(2x^2) = (2xy)^2.$$ 

However, students are first asked to prove the identity on their own in the case when $y = 1$. Very few (or likely none) of the students will use the difference of two squares identity, offering an opportunity to surprise them with the quick solution presented here.

The lesson starts with a quick review of the most important theorem in all of geometry and arguably in all of mathematics: the Pythagorean theorem. Students have already studied the Pythagorean theorem in Grade 8 and high school Geometry, have proven the theorem in numerous ways, and have used it in a wide variety of situations. Students are asked to prove it in yet a different way in the Problem Set to this lesson. The Pythagorean theorem plays an important role in both this module and the next.

Classwork

Opening Exercise (10 minutes)

This exercise is meant to help students recall facts about the Pythagorean theorem. Because it is not the main point of this lesson, feel free to move through this exercise quickly. After they have worked the problem, summarize with a statement of the Pythagorean theorem and its converse, and then move on.

Have students work in groups of two on this problem. Suggest immediately that they draw a diagram to represent the problem.

Scaffolding:

Consider starting by showing a simple example of the Pythagorean theorem.

$$5^2 + 12^2 = x^2 \Rightarrow x = 13$$
Opening Exercise
Sam and Jill decide to explore a city. Both begin their walk from the same starting point.

▪ Sam walks 1 block north, 1 block east, 3 blocks north, and 3 blocks west.
▪ Jill walks 4 blocks south, 1 block west, 1 block north, and 4 blocks east.

If all city blocks are the same length, who is the farthest distance from the starting point?

Students may have a question about what the problem is asking: Does distance mean, “Who walked the farthest?”, or “Who is the farthest (as the crow flies) from the starting point?” This question boils down to the difference between the definitions of path length versus distance. While Sam’s path length is 8 city blocks and Jill’s is 10 city blocks, the question asks for the distance between the starting point and their final destinations. To calculate distance, students need to use the Pythagorean theorem.

The problem is designed so that answers cannot be guessed easily from precisely drawn pictures.

Another (valid) issue that a student may bring up is whether the streets are considered to have width or not. Discuss this possibility with the class (again, it is a valid point). Suggest that for the purposes of this problem, the assumption is that the streets have no width (or, as some may point out, Sam and Jill could walk down the center of the streets—but this is not advisable).

Try to get students to conclude that $\sqrt{18} < \sqrt{20}$ simply because $18 < 20$ and the square root function increases.

▪ Why must the side length of a square with area 18 square units be smaller than the side length of a square with area 20 square units?
▪ Can you state the Pythagorean theorem?
  ◦ If a right triangle has legs of length $a$ and $b$ units and a hypotenuse of length $c$ units, then $a^2 + b^2 = c^2$.
▪ What is the converse of the Pythagorean theorem? Can you state it as an if–then statement?
  ◦ If the lengths $a$, $b$, $c$ of the sides of a triangle are related by $a^2 + b^2 = c^2$, then the angle opposite the side of length $c$ is a right angle.
▪ We will need the converse of the Pythagorean theorem for this lesson.
Example 1 (15 minutes)

In this example, students explore a specific case of the general method of generating Pythagorean triples, that is triples of positive integers \((a, b, c)\) that satisfy \(a^2 + b^2 = c^2\). The general form that students explore in the Problem Set is \((x^2 - y^2, 2xy, x^2 + y^2)\) for \(x > y\).

Example 1

Prove that if \(x > 1\), then a triangle with side lengths \(x^2 - 1, 2x,\) and \(x^2 + 1\) is a right triangle.

![Diagram of a right triangle with side lengths \(x^2 - 1, 2x,\) and \(x^2 + 1\).]

If \(x > 1\), is this triangle right?

Note: By the converse to the Pythagorean theorem, if \(a^2 + b^2 = c^2\), then a triangle with side lengths \(a, b, c\) is a right triangle with a right angle opposite the side of length \(c\). We are given that the triangle exists with these side lengths, so we do not need to explicitly verify that the lengths are positive. Therefore, we need only check that for any \(x > 1\), we have \((x^2 - 1)^2 + (2x)^2 = (x^2 + 1)^2\).

**Proof:** We are given a triangle with side lengths \(2x, x^2 - 1,\) and \(x^2 + 1\) for some real number \(x > 1\). We need to show that the three lengths \(2x, x^2 - 1,\) and \(x^2 + 1\) form a Pythagorean triple. We will first show that \((2x)^2\) is equivalent to \((x^2 - 1)^2 - (x^2 - 1)^2\).

\[
(x^2 + 1)^2 - (x^2 - 1)^2 = ((x^2 + 1) + (x^2 - 1))((x^2 + 1) - (x^2 - 1))
\]
\[
= (2x^2)(2)
\]
\[
= 4x^2
\]
\[
= (2x)^2
\]

Since \((2x)^2 = (x^2 + 1)^2 - (x^2 - 1)^2\), we have shown that \((x^2 - 1)^2 + (2x)^2 = (x^2 + 1)^2\), and thus the numbers \(x^2 - 1, 2x,\) and \(x^2 + 1\) form a Pythagorean triple. Then by the converse of the Pythagorean theorem, a triangle with sides of length \(2x, x^2 - 1,\) and \(x^2 + 1\) for some \(x > 1\) is a right triangle.

Proving that \((x^2 - 1)^2 + (2x)^2 = (x^2 + 1)^2\) can be done in different ways. Consider asking students to try their own method first, and then show the method above. Very few students will use the identity \(a^2 - b^2 = (a - b)(a + b)\). Most will use \((x^2 - 1)^2 + 4x^2 = x^4 - 2x^2 + 1 + 4x^2 = x^4 + 2x^2 + 1 = (x^2 + 1)^2\). This is an excellent exercise as well, since it gets students to wrestle with squares of quadratic polynomials and requires factoring. After they have tried it on their own, they will be surprised by the use of the difference of squares identity.

- A **Pythagorean triple** is a triple of positive integers \((a, b, c)\) such that \(a^2 + b^2 = c^2\). So, while \((3, 4, 5)\) is a Pythagorean triple, the triple \((1, 1, \sqrt{2})\) is not, even though \(1, 1,\) and \(\sqrt{2}\) are side lengths of a \(45^\circ-45^\circ-90^\circ\) triangle and \(1^2 + 1^2 = \left(\sqrt{2}\right)^2\). While the triangle from Example 1 can have non-integer side lengths, notice that a Pythagorean triple must comprise positive integers by definition.

- Note that any multiple of a Pythagorean triple is also a Pythagorean triple: if \((a, b, c)\) is a Pythagorean triple, then so is \((na, nb, nc)\) for any positive integer \(n\) (discuss why). Thus, \((6, 8, 10)\), \((9, 12, 15)\), \((12, 16, 20)\), \((15, 20, 25)\) are all Pythagorean triples because they are multiples of \((3, 4, 5)\).
Also note that if \((a, b, c)\) is a Pythagorean triple, then \((b, a, c)\) is also a Pythagorean triple. To reduce redundancy, we often write the smaller number of \(a\) and \(b\) first. Although \((3, 4, 5)\) and \((4, 3, 5)\) are both Pythagorean triples, they represent the same triple, and we refer to it as \((3, 4, 5)\).

One way to generate Pythagorean triples is to use the expressions from Example 1: \((x^2 - 1, 2x, x^2 + 1)\).

Have students try a few as mental math exercises: \(x = 2\) gives \((4, 3, 5)\), \(x = 3\) gives \((8, 6, 10)\), \(x = 4\) gives \((15, 8, 17)\), and so on.

One of the Problem Set questions asks students to generalize triples from \((x^2 - 1, 2x, x^2 + 1)\) to show that triples generated by \((x^2 - y^2, 2xy, x^2 + y^2)\) also form Pythagorean triples for \(x > y > 0\). The next example helps students see the general pattern.

Example 2 (12 minutes)

This example shows a clever way for students to remember that \(x^2 - 1, 2x,\) and \(x^2 + 1\) can be used to find Pythagorean triples.

Example 2

Next we describe an easy way to find Pythagorean triples using the expressions from Example 1. Look at the multiplication table below for \((1, 2, \ldots, 9)\). Notice that the square numbers \((1, 4, 9, \ldots, 81)\) lie on the diagonal of this table.

a. What value of \(x\) is used to generate the Pythagorean triple \((15, 8, 17)\) by the formula \((x^2 - 1, 2x, x^2 + 1)\)? How do the numbers \((1, 4, 9, 16, 25, 36, 49, 64, 81)\) at the corners of the shaded square in the table relate to the values 15, 8, and 17?

\[
\begin{array}{cccccccccc}
\times & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
3 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
4 & 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\
5 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
6 & 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\
7 & 7 & 14 & 21 & 28 & 35 & 42 & 48 & 55 & 63 \\
8 & 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\
9 & 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \\
10 & 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 \\
\end{array}
\]

Using the value 4 for \(x\) gives the triple \((15, 8, 17)\). We see that \(1 = 1^2\) and \(16 = 4^2\), and then we can take \(16 - 1 = 15\), and \(16 + 1 = 17\). We also have \(4 + 4 = 8\). \(\text{[}(1, 4, 8)\text{]}\)

b. Now you try one. Form a square on the multiplication table below whose left-top corner is the 1 (as in the example above) and whose bottom-right corner is a square number. Use the sums or differences of the numbers at the vertices of your square to form a Pythagorean triple. Check that the triple you generate is a Pythagorean triple.

\[
\begin{array}{cccccccccc}
\times & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\
3 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\
4 & 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\
5 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\
6 & 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\
7 & 7 & 14 & 21 & 28 & 35 & 42 & 48 & 55 & 63 \\
8 & 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\
9 & 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \\
10 & 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 \\
\end{array}
\]

Answers will vary. Ask students to report their answers. For example, a student whose square has the bottom-right number 36 will generate \(36 - 1 = 35, 6 + 6 = 12,\) and \(36 + 1 = 37\). Have students check that \((12, 35, 37)\) is indeed a Pythagorean triple: \(12^2 + 35^2 = 1369,\) and \(36^2 = 1369\).
Let's generalize this square to any square in the multiplication table where two opposite vertices of the square are square numbers.

c. How can you use the sums or differences of the numbers at the vertices of the shaded square to get a triple \((16, 30, 34)\)? Is it a Pythagorean triple?

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Following what we did above, take \(25 - 9 = 16\), \(15 + 15 = 30\), and \(25 + 9 = 34\) to get the triple \((16, 30, 34)\). Yes, it is a Pythagorean triple: \(16^2 + 30^2 = 900 + 256 = 1156 = 34^2\).

d. Using \(x\) instead of \(5\) and \(y\) instead of \(3\) in your calculations in part (c), write down a formula for generating Pythagorean triples in terms of \(x\) and \(y\).

The calculation \(25 - 9\) generalizes to \(x^2 - y^2\) as the length of one leg. The length of the other leg can be found by \(15 + 15 = 2(3 \cdot 5)\), which generalizes to \(2xy\). The length of the hypotenuse, \(25 + 9\), generalizes to \(x^2 + y^2\). It seems that Pythagorean triples can be generated by triples \((x^2 - y^2, 2xy, x^2 + y^2)\) where \(x > y > 0\).

In the Problem Set, students prove that if \(x\) and \(y\) are positive integers with \(x > y\), then \((x^2 - y^2, 2xy, x^2 + y^2)\) is a Pythagorean triple, mimicking the proof of Example 1.

Closing (3 minutes)

- Pythagorean triples are triples of positive integers \((a, b, c)\) that satisfy the relationship \(a^2 + b^2 = c^2\). Such a triple is called a Pythagorean triple because a right triangle with legs of length \(a\) and \(b\) will have a hypotenuse of length \(c\) by the Pythagorean theorem.
- To generate a Pythagorean triple, take any two positive integers \(x\) and \(y\) with \(x > y\), and compute \((x^2 - y^2, 2xy, x^2 + y^2)\).

Relevant Facts and Vocabulary

**PYTHAGOREAN THEOREM:** If a right triangle has legs of length \(a\) and \(b\) units and hypotenuse of length \(c\) units, then \(a^2 + b^2 = c^2\).

**CONVERSE TO THE PYTHAGOREAN THEOREM:** If the lengths \(a, b, c\) of the sides of a triangle are related by \(a^2 + b^2 = c^2\), then the angle opposite the side of length \(c\) is a right angle.

**PYTHAGOREAN TRIPLE:** A Pythagorean triple is a triple of positive integers \((a, b, c)\) such that \(a^2 + b^2 = c^2\). The triple \((3, 4, 5)\) is a Pythagorean triple but \((1, 1, \sqrt{2})\) is not, even though the numbers are side lengths of an isosceles right triangle.

Exit Ticket (5 minutes)
Lesson 10: The Power of Algebra—Finding Pythagorean Triples

Exit Ticket

Generate six Pythagorean triples using any method discussed during class. Explain each method you use.
Exit Ticket Sample Solutions

Generate six Pythagorean triples using any method discussed during class. Explain each method you use.

Answers will vary. One example should use either \((x^2 - 1, 2x, x^2 + 1)\) or \((x^2 - y^2, 2xy, x^2 + y^2)\), but after that students can use the fact that a multiple of a Pythagorean triple is again a Pythagorean triple.

Problem Set Sample Solutions

1. Rewrite each expression as a sum or difference of terms.
   a. \((x - 3)(x + 3)\)  
      \(x^2 - 9\)
   b. \((x^2 - 3)(x^2 + 3)\)
      \(x^4 - 9\)
   c. \((x^{15} + 3)(x^{15} - 3)\)
      \(x^{30} - 9\)
   d. \((x - 3)(x^2 + 9)(x + 3)\)
      \(x^4 - 81\)
   e. \((x^2 + y^2)(x^2 - y^2)\)
      \(x^4 - y^4\)
   f. \((x^2 + y^2)^2\)
      \(x^4 + 2x^2y^2 + y^4\)
   g. \((x - y)^2(x + y)^2\)
      \(x^4 - 2x^2y^2 + y^4\)
   h. \((x - y)^2(x^2 + y^2)(x + y)^2\)
      \(x^8 - 2x^4y^4 + y^8\)

2. Tasha used a clever method to expand \((a + b + c)(a + b - c)\). She grouped the addends together like this \([[(a + b) + c][(a + b) - c]]\) and then expanded them to get the difference of two squares:
   \[(a + b + c)(a + b - c) = [(a + b) + c][(a + b) - c] = (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2.\]
   a. Is Tasha’s method correct? Explain why or why not.
      Yes, Tasha is correct. Expanding in the traditional way gives the same result.
      \[(a + b + c)(a + b - c) = (a + b + c)a + (a + b + c)b - (a + b + c)c\]
      \[= a^2 + ba + ca + ab + b^2 + cb - ac - bc - c^2\]
      \[= a^2 + 2ab + b^2 - c^2\]
   b. Use a version of her method to find \((a + b + c)(a - b - c)\).
      \[(a + (b + c))(a - (b + c)) = a^2 - (b + c)^2 = a^2 - b^2 - 2bc - c^2\]
   c. Use a version of her method to find \((a + b - c)(a - b + c)\).
      \[(a + (b - c))(a - (b + c)) = a^2 - (b - c)^2 = a^2 - b^2 + 2bc - c^2\]

3. Use the difference of two squares identity to factor each of the following expressions.
   a. \(x^2 - 81\)
      \((x - 9)(x + 9)\)
   b. \((3x + y)^2 - (2y)^2\)
      \((3x - y)(3x + 3y) = 3(3x - y)(x + y)\)
   c. \(4 - (x - 1)^2\)
      \((3 - x)(1 + x)\)
   d. \((x + 2)^2 - (y + 2)^2\)
      \((x - y)(x + y + 4)\)
4. Show that the expression \((x + y)(x - y) - 6x + 9\) may be written as the difference of two squares, and then factor the expression.

\[(x + y)(x - y) - 6x + 9 = x^2 - y^2 - 6x + 9 = (x^2 - 6x + 9) - y^2 = (x - 3)^2 - y^2 = (x - 3 - y)(x - 3 + y)\]

5. Show that \((x + y)^2 - (x - y)^2 = 4xy\) for all real numbers \(x\) and \(y\).

\[(x + y)^2 - (x - y)^2 = [(x + y) - (x - y)][(x + y) + (x - y)] = (2y)(2x) = 4xy\]

6. Prove that a triangle with side lengths \(x^2 - y^2, 2xy,\) and \(x^2 + y^2\) with \(x > y > 0\) is a right triangle.

*The proof should look like the proof in Example 1 but with \(y\) instead of 1.*

7. Complete the table below to find Pythagorean triples (the first row is done for you).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(x^2 - y^2)</th>
<th>(2xy)</th>
<th>(x^2 + y^2)</th>
<th>Check: Is it a Pythagorean Triple?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>Yes: (3^2 + 4^2 = 25 = 5^2)</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>8</td>
<td>6</td>
<td>10</td>
<td>Yes: (8^2 + 6^2 = 100 = 10^2)</td>
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<tr>
<td>3</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>13</td>
<td>Yes: (5^2 + 12^2 = 169 = 13^2)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>15</td>
<td>8</td>
<td>17</td>
<td>Yes: (15^2 + 8^2 = 289 = 17^2)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>Yes: (12^2 + 16^2 = 400 = 20^2)</td>
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<tr>
<td>4</td>
<td>3</td>
<td>7</td>
<td>24</td>
<td>25</td>
<td>Yes: (7^2 + 24^2 = 625 = 25^2)</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>24</td>
<td>10</td>
<td>26</td>
<td>Yes: (24^2 + 10^2 = 676 = 26^2)</td>
</tr>
</tbody>
</table>

8. Answer the following parts about the triple \((9, 12, 15)\).

a. Show that \((9, 12, 15)\) is a Pythagorean triple.

*We see that \(9^2 + 12^2 = 81 + 144 = 225\), and \(15^2 = 225\) so \(9^2 + 12^2 = 15^2\).*

b. Prove that neither \((9, 12, 15)\) nor \((12, 9, 15)\) can be found by choosing a pair of integers \(x\) and \(y\) with \(x > y\) and computing \((x^2 - y^2, 2xy, x^2 + y^2)\).

*Hint: What are the possible values of \(x\) and \(y\) if \(2xy = 12\)? What about if \(2xy = 9\)?*

*Proof: Since \(9\) is odd and \(2xy\) is even, there are no integer values of \(x\) and \(y\) that satisfy \(2xy = 9\). Thus, our formula cannot generate the triple \((12, 9, 15)\). Now suppose \(x\) and \(y\) are integers such that \(2xy = 12\). Thus \(xy = 6\) and \(x > y\). There are only two possibilities: either \(x = 6\) and \(y = 1\), or \(x = 3\) and \(y = 2\). In the first case, our formula generates the triple \((6^2 - 1, 2 \cdot 6 - 1, 6^2 + 1) = (35, 12, 37)\). In the second case, our formula generates the triple \((3^2 - 2^2, 2 \cdot 3 - 2, 3^2 + 2^2) = (5, 12, 13)\). Thus, there is no way to generate the triple \((9, 12, 15)\) using this method, even though it is a Pythagorean triple.*

c. Wouldn’t it be nice if all Pythagorean triples were generated by \((x^2 - y^2, 2xy, x^2 + y^2)\)? Research Pythagorean triples on the Internet to discover what is known to be true about generating all Pythagorean triples using this formula.

*All Pythagorean triples are some multiple of a Pythagorean triple generated using this formula. For example, while \((9, 12, 15)\) is not generated by the formula, it is a multiple of a Pythagorean triple \((3, 4, 5)\), which is generated by the formula.*

9. Follow the steps below to prove the identity \((a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2\).

a. Multiply \((a^2 + b^2)(x^2 + y^2)\).

\[(a^2 + b^2)(x^2 + y^2) = a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2\]
b. Square both binomials in \((ax - by)^2 + (bx + ay)^2\) and collect like terms.
\[
(ax - by)^2 + (bx + ay)^2 = a^2x^2 - 2axby + b^2y^2 + b^2x^2 + 2axby + a^2y^2
\]
\[
= a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2
\]

c. Use your answers from part (a) and part (b) to prove the identity.
\[
(a^2 + b^2)(a^2 + b^2) = a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2
\]
\[
= (ax - by)^2 + (bx + ay)^2
\]

10. Many U.S. presidents took great delight in studying mathematics. For example, President James Garfield, while still a congressman, came up with a proof of the Pythagorean theorem based upon the ideas presented below.

In the diagram, two congruent right triangles with side lengths \(a\), \(b\), and hypotenuse \(c\), are used to form a trapezoid \(PQRS\) composed of three triangles.

a. Explain why \(\angle QTR\) is a right angle.

Since \(\angle TSR\) is a right angle, the measures of \(\angle STR\) and \(\angle SRT\) sum to 90°, so \(\angle STR\) and \(\angle SRT\) are complementary angles. Since \(\triangle TSR \cong \triangle QPT\) by SSS triangle congruence, we have \(\angle STR \cong \angle PTQ\). Thus, \(\angle PTQ\) and \(\angle STR\) must also be complementary. By the angle sum properties,
\[
m\angle QTR + m\angle PTQ + m\angle STR = 180°
\]
so that
\[
m\angle QTR + 90° = 180°
\]
and we have shown that \(m\angle QTR = 90°\). Thus, \(\angle QTR\) is a right angle.

b. What are the areas of \(\triangle STR\), \(\triangle PTQ\), and \(\triangle QTR\) in terms of \(a\), \(b\), and \(c\)?

We see that \(A(\triangle STR) = \frac{1}{2}ab\), \(A(\triangle PTQ) = \frac{1}{2}ab\), and because \(\angle QTR\) is a right angle, \(A(\angle QTR) = \frac{1}{2}c^2\).

c. Using the formula for the area of a trapezoid, what is the total area of trapezoid \(PQRS\) in terms of \(a\) and \(b\)?

\[
A(PQRS) = \frac{1}{2}(a + b)(a + b)
\]

d. Set the sum of the areas of the three triangles from part (b) equal to the area of the trapezoid you found in part (c), and simplify the equation to derive a relationship between \(a\), \(b\), and \(c\). Conclude that a right triangle with legs of length \(a\) and \(b\) and hypotenuse of length \(c\) must satisfy the relationship \(a^2 + b^2 = c^2\).

Equate areas:
\[
\frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a + b)(a + b),
\]
\[
ab + \frac{1}{2}c^2 = \frac{1}{2}(a^2 + 2ab + b^2).
\]

Multiply both sides by 2,
\[
2ab + c^2 = a^2 + 2ab + b^2,
\]

and subtract \(2ab\) from both sides,
\[
c^2 = a^2 + b^2.
\]