Lesson 21: Comparing Linear and Exponential Models Again

Student Outcomes

- Students create models and understand the differences between linear and exponential models that are represented in different ways.

Lesson Notes

Students have spent several lessons in this module studying linear and exponential functions. This lesson has students compare and contrast the two functions to solidify their understanding of each. Students then learn how to differentiate between the two functions by recognizing a constant rate of change or a common quotient in data tables or by recognizing the features in their graphs or formulas.

Classwork

Opening Exercise (10 minutes)

This table format allows students to compare linear and exponential models in several ways. This exercise reminds students what linear and exponential models are and allows them to easily see the difference between them. Students can either be given the example or write their own for each model. Once students have finished, have them discuss their work in pairs or in groups. Tie this together with a class discussion, paying particular attention to the different wordings that students use and the real-world examples that they give. Like they did in the prior lesson, students use MP.7 to identify the meaning of structural components of each function type.

<table>
<thead>
<tr>
<th>Opening Exercise</th>
<th>Linear Model</th>
<th>Exponential Model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General Form</strong></td>
<td>( f(x) = ax + b )</td>
<td>( f(x) = a(b)^x )</td>
</tr>
<tr>
<td><strong>Meaning of Parameters</strong></td>
<td>( a ) is the slope of the line or the constant rate of change; ( b ) is the ( y )-intercept or the ( f(x) ) value at ( x = 0 ).</td>
<td>( a ) is the ( y )-intercept or the ( f(x) ) value when ( x = 0 ); ( b ) is the base or the constant quotient of change.</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>( f(x) = 2x + 3 )</td>
<td>( f(x) = 3(2)^x )</td>
</tr>
<tr>
<td><strong>Rule for Finding ( f(x + 1) ) from ( f(x) )</strong></td>
<td>( \text{Starting at (0, 3), to find } f(x + 1), \text{ add 2 to } f(x). )</td>
<td>( \text{Starting at (0, 3), to find } f(x + 1), \text{ multiply } f(x) \text{ by 2.} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table</th>
<th>Linear Model</th>
<th>Exponential Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f(x) )</td>
<td>( f(x) )</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

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In these exercises, students complete the table for the function \( f \) and find a progression of differences of two outputs of \( f \) for two inputs that have a difference of 1 unit (i.e., they find the change in \( f \), \( \Delta f = f(x + 1) - f(x) \), where the difference between \( x + 1 \) and \( x \) is 1). If \( \Delta f = f(x + 1) - f(x) \) is constant for all input values \( x \) and \( x + 1 \) in the table, students identify the function given by the table as potentially linear. Similarly, when the quotient \( \frac{f(x+1)}{f(x)} \) is a constant for all input values \( x \) and \( x + 1 \), they identify the function as potentially exponential. This activity can be done individually or in groups. Upon completion, a class discussion should take place.

Be sure that students understand the following telltale signs for recognizing linear and exponential functions from a table of input-output pairs: For each \( d > 0 \), consider all inputs that have a difference of \( d \) units:

- If the difference between their corresponding outputs is always the same constant, then the input-output pairs in the table can be modeled by a linear function;
- If the quotient between their corresponding outputs is always the same constant, then the input-output pairs in the table can be modeled by an exponential function.

In the exercise below, \( d = 1 \) is used, but point out to students that these statements also make sense for \( d = 2, d = 3 \), etc.

### Exercises (12 minutes)

1. For each table below, assume the function \( f \) is defined for all real numbers. Calculate \( \Delta f = f(x + 1) - f(x) \) in the last column in the tables below, and show your work. (The symbol \( \Delta \) in this context means change in.) What do you notice about \( \Delta f \)? Could the function be linear or exponential? Write a linear or an exponential function formula that generates the same input-output pairs as given in the table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f = f(x + 1) - f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3</td>
<td>( 1 - (-3) = 4 )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( 5 - 1 = 4 )</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>( 9 - 5 = 4 )</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>( 13 - 9 = 4 )</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

\( f(x) = 4x - 7 \)
Students should see that \( \Delta f \) is constant for any two inputs that have a difference of 1 unit, which implies that the function could be a linear function.

Point out that there are infinitely many functions that map each input in the table to the corresponding output, but only one of them is linear, and none of them can be exponential.

In this example (and other linear examples like it), the constant change in \( f \) depends on the choice of \( d \). However, the rate of change in \( f \) per the distance \( d \) is always the constant 4; that is, \( \frac{f(x+d)−f(x)}{d} = 4 \) for all \( x \) and \( d \). This constant rate of change is the slope of the line.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f = f(x+1)−f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>6−2 = 4</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>18−6 = 12</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>54−18 = 36</td>
</tr>
<tr>
<td>3</td>
<td>54</td>
<td>162−54 = 108</td>
</tr>
<tr>
<td>4</td>
<td>162</td>
<td></td>
</tr>
</tbody>
</table>

If the entries in the table were considered as a geometric sequence, then the common quotient would be \( r = 3 \). Since \( f(0) = 2, a = 2 \). Since \( f(1) = 6 \), we must have \( 6 = 2 \cdot b \) or \( b = 3 \). Hence, \( f(x) = 2(3)^x \).

In this table, students should see that \( \Delta f \) is not constant for any two inputs that have a difference of 1 unit, which implies that the function cannot be a linear function. However, there is a common quotient between inputs that have a difference of 1 unit: \( \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = \frac{162}{54} \). Hence, the function \( f \) could be exponential.

Again, point out that there are infinitely many functions that map each input in the table to the corresponding output, but none of them can be linear, and only one of them can be of the form \( f(x) = ab^x \) (exponential).

2. Terence looked down the second column of the table below and noticed that \( \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \frac{81}{27} \). Because of his observation, he claimed that the input-output pairs in this table could be modeled with an exponential function. Explain why Terence is correct or incorrect. If he is correct, write a formula for the exponential function that generates the input-output pairs given in the table. If he is incorrect, determine and write a formula for a function that generates the input-output pairs given in the table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( T(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>13</td>
<td>27</td>
</tr>
<tr>
<td>40</td>
<td>81</td>
</tr>
</tbody>
</table>

Terence is incorrect. This is because the distance between consecutive \( x \)-values is not 1 (not constant).
Some students may claim they are correct and use the first two input-output pairs to get the formula $T(x) = 3^x$. Ask them to see if it satisfies the remaining input-output pairs. Guide (hint to) students what function models this table by asking them to compute the average rate of change (this term may need to be recalled for them) between input-output pairs. If they do, they quickly see that the average rate of change is always 2. They should then be able to quickly derive the linear function $T(x) = 2x + 1$ using the first two input-output pairs. Encourage them to check their answers.

In groups, have students look at Exercises 3 and 4. After groups have had time to discuss their solutions, have them discuss their results as a class. The purpose of these exercises is to help students understand when a situation models exponential or linear growth or decay. Exercise 4 reemphasizes that the value of an exponential function with a base greater than 1 always exceeds any linear function for large enough positive input numbers. Exercise 4 highlights MP.3 as it calls on students to evaluate a hypothetical claim.

Exercises 3–4 (14 minutes)

3. A river has an initial minnow population of 40,000 that is growing at 5% per year. Due to environmental conditions, the amount of algae that minnows use for food is decreasing, supporting 1,000 fewer minnows each year. Currently, there is enough algae to support 50,000 minnows. Is the minnow population increasing linearly or exponentially? Is the amount of algae decreasing at a linear or an exponential rate? In what year will the minnow population exceed the amount of algae available?

The minnow population is increasing exponentially. The amount of algae is decreasing the food supply that can be used to support minnows linearly. Solution methods to finding when the minnow population will exceed the amount of algae may vary. Here is one solution: We need to find the number of years $n$ such that $40(1.05)^n > 50 - n$. (Why?) For $n = 3$, we get $46.3 > 47$ is false, but for $n = 4$, we get that $48.6 > 46$ is true. Hence, some time after the third year, the minnow population will exceed the amount of algae available.

4. Using a calculator, Joanna made the following table and then made the following conjecture: $3x$ is always greater than $(1.02)^x$. Is Joanna correct? Explain.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$(1.02)^x$</th>
<th>$3x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.02</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1.0404</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>1.0612</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>1.0824</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>1.1041</td>
<td>15</td>
</tr>
</tbody>
</table>

The function $(1.02)^x$ is exponential, and the function $3x$ is linear, so the exponential will eventually exceed the linear. (Alternatively, students could extend the table to larger values of $x$ to show that the exponential function will eventually exceed the linear function.) $(1.02)^{500} = 19957$, but $3(500) = 1500$
Closing (4 minutes)

- How can you tell whether input-output pairs in a table are describing a linear relationship or an exponential relationship?

Lesson Summary

- Suppose that the input-output pairs of a bivariate data set have the following property: For every two inputs that are a given difference apart, the difference in their corresponding outputs is constant. Then, an appropriate model for that data set could be a linear function.

- Suppose that the input-output pairs of a bivariate data set have the following property: For every two inputs that are a given difference apart, the quotient of their corresponding outputs is constant. Then, an appropriate model for that data set could be an exponential function.

- An increasing exponential function will eventually exceed any linear function. That is, if \( f(x) = ab^x \) is an exponential function with \( a > 0 \) and \( b > 1 \), and \( g(x) = mx + k \) is any linear function, then there is a real number \( M \) such that for all \( x > M \), then \( f(x) > g(x) \). Sometimes this is not apparent in a graph displayed on a graphing calculator; that is because the graphing window does not show enough of the graph to show the sharp rise of the exponential function in contrast with the linear function.

Exit Ticket (5 minutes)
Exit Ticket

Here is a classic riddle: Mr. Smith has an apple orchard. He hires his daughter, Lucy, to pick apples and offers her two payment options:

Option A: $1.50 per bushel of apples picked

Option B: 1 cent for picking one bushel, 3 cents for picking two bushels, 9 cents for picking three bushels, and so on, with the amount of money tripling for each additional bushel picked

a. Write a function to model each option.

b. If Lucy picks six bushels of apples, which option should she choose?

c. If Lucy picks 12 bushels of apples, which option should she choose?

d. How many bushels of apples does Lucy need to pick to make Option B better for her than Option A?
Exit Ticket Sample Solutions

Here is a classic riddle: Mr. Smith has an apple orchard. He hires his daughter, Lucy, to pick apples and offers her two payment options:

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Option B: 1 cent for picking one bushel, 3 cents for picking two bushels, 9 cents for picking three bushels, and so on, with the amount of money tripling for each additional bushel picked

a. Write a function to model each option.

Option A: \( f(x) = 1.50x \), where \( x \) is the number of bushels of apples picked

Option B: \( g(x) = 0.01(3)^{x-1} \), where \( x \) is the number of bushels of apples picked

b. If Lucy picks six bushels of apples, which option should she choose?

Option A: $9  
Option B: $2.43  
Option A is better.

If Lucy picks 12 bushels of apples, which option should she choose?

Option A: $18  
Option B: $1,771.47  
Option B is much better.

c. How many bushels of apples does Lucy need to pick to make Option B better for her than Option A?

The eighth bushel picked is when the exponential function exceeds the linear function.

Problem Set Sample Solutions

For each table in Problems 1–6, classify the data as describing a linear relationship, an exponential growth relationship, an exponential decay relationship, or neither. If the relationship is linear, calculate the constant rate of change (slope), and write a formula for the linear function that models the data. If the function is exponential, calculate the common quotient for input values that are distance one apart, and write the formula for the exponential function that models the data. For each linear or exponential function found, graph the equation \( y = f(x) \).

1. \[
\begin{array}{c|c}
  x & f(x) \\
  
  1 & \frac{1}{2} \\
  2 & \frac{1}{4} \\
  3 & \frac{1}{8} \\
  4 & \frac{1}{16} \\
  5 & \frac{1}{32} \\
\end{array}
\]

Common quotient: \( \frac{1}{2} \), exponential decay, \( f(x) = \left(\frac{1}{2}\right)^x \)
### Lesson 21: Comparing Linear and Exponential Models Again

#### 2. 
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
</tr>
<tr>
<td>3</td>
<td>3.6</td>
</tr>
<tr>
<td>4</td>
<td>4.7</td>
</tr>
<tr>
<td>5</td>
<td>5.8</td>
</tr>
</tbody>
</table>

**Constant rate of change: 1.1, linear,** $f(x) = 0.3 + 1.1x$

#### 3. 
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

**Neither**

#### 4. 
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
</tr>
<tr>
<td>4</td>
<td>160</td>
</tr>
<tr>
<td>5</td>
<td>320</td>
</tr>
</tbody>
</table>

**Ratio of change: 2, exponential growth,** $f(x) = 10(2)^x$
5. \[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
1 & -5 \\
2 & -12 \\
3 & -19 \\
4 & -26 \\
5 & -33 \\
\hline
\end{array}
\]

Rate of change: $-7$, linear, $f(x) = 2 - 7x$

6. \[
\begin{array}{|c|c|}
\hline
x & f(x) \\
\hline
1 & \frac{1}{2} \\
2 & \frac{1}{3} \\
3 & \frac{1}{4} \\
4 & \frac{1}{5} \\
5 & \frac{1}{6} \\
\hline
\end{array}
\]

Neither

7. Here is a variation on a classic riddle: Jayden has a dog-walking business. He has two plans. Plan 1 includes walking a dog once a day for a rate of $5 per day. Plan 2 also includes one walk a day but charges 1 cent for 1 day, 2 cents for 2 days, 4 cents for 3 days, and 8 cents for 4 days and continues to double for each additional day. Mrs. Maroney needs Jayden to walk her dog every day for two weeks. Which plan should she choose? Show the work to justify your answer.

Plan 1: $5 \times 14 = 70$ She will pay $70 for two weeks under Plan 1.

Plan 2: The price can be modeled using the function $f(x) = 0.01(2)^{x-1}$, which describes the amount she pays for $x$ days. Then,

$f(14) = 0.01(2)^{14-1} = 81.92$.

She will pay $81.92 under Plan 2 for two weeks.

Plan 1 is the better choice.

8. Tim deposits money in a certificate of deposit account. The balance (in dollars) in his account $t$ years after making the deposit is given by $T(t) = 1000(1.06)^t$ for $t \geq 0$.

a. Explain, in terms of the structure of the expression used to define $T(t)$, why Tim’s balance can never be $999$.

$(1.06)^0 = 1$, and positive powers of $1.06$ are larger than $1$; thus, the minimum value $T(t)$ attains, if $t \geq 0$, is $1,000$. In the context given, a CD account grows in value over time, so with a deposit of $1,000$, the value will never drop to $999$. 
b. By what percent does the value of \( T(t) \) grow each year? Explain by writing a recursive formula for the sequence \( T(1), \ T(2), \ T(3), \) etc.

\[
\text{Writing out the sequence, we see:}
\]
Year 1: \( 1000(1.06)^1 \)
Year 2: \( 1000(1.06)^2 \)
Year 3: \( 1000(1.06)^3 \)

\[
\text{Thus, } T(n + 1) = T(n)(1.06) \text{ or } T(n + 1) = T(n) + T(n)(0.06), \text{ showing that } T(n) \text{ grows by 6\% per year.}
\]

c. By what percentage does the value of \( T(t) \) grow every two years? (Hint: Use your recursive formula to write \( T(n + 2) \) in terms of \( T(n) \).)

\[
\text{Since, } T(n + 1) = T(n)(1.06), \text{ we can write:}
\]
\[
T(n + 2) = T(n + 1)(1.06)
= T(n)(1.06)(1.06),
\]
or \( T(n + 2) = T(n)(1.1236). \)

\[
\text{Hence, the amount in his account grows by 12.36\% every two years.}
\]

9. Your mathematics teacher asks you to sketch a graph of the exponential function \( f(x) = \left(\frac{3}{2}\right)^{x} \) for \( x \), a number between 0 and 40 inclusively, using a scale of 10 units to one inch for both the \( x- \) and \( y- \) axes.

a. What are the dimensions (in feet) of the roll of paper needed to sketch this graph?

\[
\text{The roll would need to be a bit wider than 4 in. \( \left(\frac{1}{3} \text{ ft.}\right) \) and, after rounding up, about 1,105,734 in. (or 92.145 ft. long).}
\]

b. How many more feet of paper would you need to add to the roll in order to graph the function on the interval \( 0 \leq x \leq 41 \)?

\[
\text{It would require approximately 46,072 more feet of paper. Teachers: Look for the easy solution, which is}
\]
\[
f(41) = f(40) \times 1.5 = f(40) + f(40) \times \frac{1}{2},
\]

c. Find an \( m \) so that the linear function \( g(x) = mx + 2 \) is greater than \( f(x) \) for all \( x \) such that \( 0 \leq x \leq 40 \), but \( f(41) > g(41) \).

\[
\text{There are many possible answers: Any number between (roughly) 276,433.275 and 404,536 will do. Note that} f(40) \approx 11 \, 057 \, 332.32 \text{ and } f(41) \approx 16 \, 585 \, 998.48, \text{ so any slope such that } 11 \, 057 \, 333 \leq g(40) \text{ and } g(41) \leq 16 \, 585 \, 998 \text{ will do. Since } g(40) = 40m + 2, \text{ solving } 11 \, 057 \, 333 = 40m + 2 \text{ or} m = 276 \, 433.275 \text{ might produce such a linear function. We need to check only that the function}
\]
\[
g(x) = 276 \, 433.275x + 2 \text{ satisfies } f(41) > g(41), \text{ which it does because } g(41) = 11 \, 333 \, 766.275.
\]