Lesson 20: Four Interesting Transformations of Functions

Student Outcomes
- Students apply their understanding of transformations of functions and their graphs to piecewise functions.

Lesson Notes
In Lessons 17–19, students study translations and scalings of functions and their graphs. In this lesson, these transformations are applied in combination with piecewise functions. Students become comfortable visualizing how the graph of a transformed piecewise function relates to the graph of the original piecewise function.

Classwork
Opening Exercise (6 minutes)

Have students work individually or in pairs to complete the Opening Exercise. This exercise highlights MP.7 since it calls on students to interpret the meaning of \( k \) in the context of a graph.

<table>
<thead>
<tr>
<th>Graph of ( y = f(x) )</th>
<th>Vertical</th>
<th>Horizontal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translate ( y = f(x) + k )</td>
<td>( k &gt; 0 )</td>
<td>Translate up by</td>
</tr>
<tr>
<td></td>
<td>( k &lt; 0 )</td>
<td>(</td>
</tr>
<tr>
<td>Scale by scale factor ( k ) ( y = kf(x) )</td>
<td>( k &gt; 1 )</td>
<td>Vertical stretch by a factor of (</td>
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<td></td>
<td>( 0 &lt; k &lt; 1 )</td>
<td>Vertical shrink by a factor of (</td>
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<td></td>
<td>( -1 &lt; k &lt; 0 )</td>
<td>Vertical shrink by a factor of (</td>
</tr>
<tr>
<td></td>
<td>( k &lt; -1 )</td>
<td>Vertical stretch by a factor of (</td>
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</tbody>
</table>
In Lesson 15, students discovered how the absolute value function can be written as a piecewise function. Exploratory Challenge 1 and the associated exercises are intended to help students reexamine how piecewise functions behave.

Exploratory Challenge 1 (3 minutes)

Exploratory Challenge 1
A transformation of the absolute value function \( f(x) = |x - 3| \) is rewritten here as a piecewise function. Describe in words how to graph this piecewise function.

\[
f(x) = \begin{cases} 
-x + 3, & x < 3 \\
x - 3, & x \geq 3 
\end{cases}
\]

First, I would graph the line \( y = -x + 3 \) for \( x \)-values less than 3, and then I would graph the line \( y = x - 3 \) for \( x \)-values greater than or equal to 3.

Exercises 1–2 (15 minutes)

1. Describe how to graph the following piecewise function. Then, graph \( y = f(x) \) below.

\[
f(x) = \begin{cases} 
-3x - 3, & x \leq -2 \\
0.5x + 4, & -2 < x < 2 \\
-2x + 9, & x \geq 2 
\end{cases}
\]

The function \( f \) can be graphed of the line \( y = -3x - 3 \) for \( x \)-values less than or equal to \(-2\), the graph of the line \( y = 0.5x + 4 \) for \( x \)-values greater than \(-2\) and less than \(2\), and the graph of the line \( y = -2x + 9 \) for \( x \)-values greater than or equal to \(2\).
2. Using the graph of \( f \) below, write a formula for \( f \) as a piecewise function.

\[
f(x) = \begin{cases} 
0.5x - 0.5, & -7 \leq x \leq -3 \\
-2, & -3 < x < 1 \\
x - 3, & 1 \leq x \leq 4 \\
5 - x, & 4 < x \leq 7 
\end{cases}
\]

OR

\[
f(x) = \begin{cases} 
0.5x - 0.5, & -7 \leq x \leq -3 \\
-2, & -3 < x < 1 \\
-|x - 4| + 1, & 1 \leq x \leq 7 
\end{cases}
\]

Exploratory Challenge 2 (10 minutes)

Students translate and scale the graph of a piecewise function.

Exploratory Challenge 2

The graph \( y = f(x) \) of a piecewise function \( f \) is shown. The domain of \( f \) is \(-5 \leq x \leq 5\), and the range is \(-1 \leq y \leq 3\).

a. Mark and identify four strategic points helpful in sketching the graph of \( y = f(x) \).

\((-5, -1), (-1, 1), (3, 1), \text{and} (5, 3)\)
b. Sketch the graph of \( y = 2f(x) \), and state the domain and range of the transformed function. How can you use part (a) to help sketch the graph of \( y = 2f(x) \)?

**Domain:** \(-5 \leq x \leq 5\), **range:** \(-2 \leq y \leq 6\). For every point \((x, y)\) on the graph of \( y = f(x) \), there is a point \((x, 2y)\) on the graph of \( y = 2f(x) \). The four strategic points can be used to determine the line segments in the graph of \( y = 2f(x) \) by graphing points with the same original \( x \)-coordinate and 2 times the original \( y \)-coordinate \((-5, -2), (-1, 2), (3, 2), \) and \((5, 6)\).

![Graph of \( y = f(x) \) and \( y = 2f(x) \)](image)

b. A horizontal scaling with scale factor \( \frac{1}{2} \) of the graph of \( y = f(x) \) is the graph of \( y = f(2x) \). Sketch the graph of \( y = f(2x) \), and state the domain and range. How can you use the points identified in part (a) to help sketch \( y = f(2x) \)?

**Domain:** \(-2.5 \leq x \leq 2.5\), **range:** \(-1 \leq y \leq 3\). For every point \((x, y)\) on the graph of \( y = f(x) \), there is a point \((\frac{x}{2}, y)\) on the graph of \( y = f(2x) \). The four strategic points can be used to determine the line segments in the graph of \( y = f(2x) \) by graphing points with one-half the original \( x \)-coordinate and the original \( y \)-coordinate \((-2.5, -1), (-0.5, 1), (1.5, 1), \) and \((2.5, 3)\).

c. A horizontal scaling with scale factor \( \frac{1}{2} \) of the graph of \( y = f(x) \) is the graph of \( y = f(2x) \). Sketch the graph of \( y = f(2x) \), and state the domain and range. How can you use the points identified in part (a) to help sketch \( y = f(2x) \)?

**Domain:** \(-2.5 \leq x \leq 2.5\), **range:** \(-1 \leq y \leq 3\). For every point \((x, y)\) on the graph of \( y = f(x) \), there is a point \((\frac{x}{2}, y)\) on the graph of \( y = f(2x) \). The four strategic points can be used to determine the line segments in the graph of \( y = f(2x) \) by graphing points with one-half the original \( x \)-coordinate and the original \( y \)-coordinate \((-2.5, -1), (-0.5, 1), (1.5, 1), \) and \((2.5, 3)\).
Exercises 3–4 (5 minutes)

3. How does the range of \( f \) in Exploratory Challenge 2 compare to the range of a transformed function \( g \), where \( g(x) = kf(x) \), when \( k > 1 \)?

   For every point \((x, y)\) in the graph of \( y = f(x) \), there is a point \((x, ky)\) in the graph of \( y = kf(x) \), where the number \( ky \) is a multiple of each \( y \). For values of \( k > 1 \), \( y = kf(x) \) is a vertical scaling that appears to stretch the graph of \( y = f(x) \). The original range, \(-1 \leq y \leq 3\) for \( y = f(x) \) becomes \(-1k \leq y \leq 3k\) for the function \( y = kf(x) \).

4. How does the domain of \( f \) in Exploratory Challenge 2 compare to the domain of a transformed function \( g \), where \( g(x) = f\left(\frac{1}{k}x\right) \), when \( 0 < k < 1 \)? (Hint: How does a graph shrink when it is horizontally scaled by a factor \( k \)?)

   For every point \((x, y)\) in the graph of \( y = f(x) \), there is a point \((kx, y)\) in the graph of \( y = f\left(\frac{1}{k}x\right) \). For values of \( 0 < k < 1 \), \( y = f\left(\frac{1}{k}x\right) \) is a horizontal scaling by a factor \( k \) that appears to shrink the graph of \( y = f(x) \).

This means the original domain, \(-5 \leq x \leq 5\) for \( y = f(x) \), becomes \(-5k \leq x \leq 5k\) for the function \( y = f\left(\frac{1}{k}x\right) \).

Closing (2 minutes)

- The transformations that translate and scale familiar functions, like the absolute value function, also apply to piecewise functions and to any function in general.
- By focusing on strategic points in the graph of a piecewise function, the entire graph of the function can be translated and scaled by manipulating the coordinates of those few points.

Exit Ticket (4 minutes)
Lesson 20: Four Interesting Transformations of Functions

Exit Ticket

The graph of a piecewise function \( f \) is shown below.

Let \( p(x) = f(x - 2) \), \( q(x) = \frac{1}{2} f(x - 2) \), and \( r(x) = \frac{1}{2} f(x - 2) + 3 \).

Graph \( y = p(x), y = q(x) \), and \( y = r(x) \) on the same set of axes as the graph of \( y = f(x) \).
Exit Ticket Sample Solutions

The graph of a piecewise function $f$ is shown below.

Let $p(x) = f(x - 2)$, $q(x) = \frac{1}{2} f(x - 2)$, and $r(x) = \frac{1}{2} f(x - 2) + 3$.

Graph $y = p(x)$, $y = q(x)$, and $y = r(x)$ on the same set of axes as the graph of $y = f(x)$. 
Problem Set Sample Solutions

1. Suppose the graph of $f$ is given. Write an equation for each of the following graphs after the graph of $f$ has been transformed as described. Note that the transformations are not cumulative.
   a. Translate 5 units upward.
      \[ y = f(x) + 5 \]
   b. Translate 3 units downward.
      \[ y = f(x) - 3 \]
   c. Translate 2 units right.
      \[ y = f(x - 2) \]
   d. Translate 4 units left.
      \[ y = f(x + 4) \]
   e. Reflect about the $x$-axis.
      \[ y = -f(x) \]
   f. Reflect about the $y$-axis.
      \[ y = f(-x) \]
   g. Stretch vertically by a factor of 2.
      \[ y = 2f(x) \]
   h. Shrink vertically by a factor of $\frac{1}{3}$.
      \[ y = \frac{1}{3}f(x) \]
   i. Shrink horizontally by a factor of $\frac{1}{3}$.
      \[ y = f(3x) \]
   j. Stretch horizontally by a factor of 2.
      \[ y = f\left(\frac{1}{2}x\right) \]

2. Explain how the graphs of the equations below are related to the graph of $y = f(x)$.
   a. $y = 5f(x)$
      The graph is a vertical stretch of $y = f(x)$ by a factor of 5.
b. \( y = f(x - 4) \)  
The graph of \( y = f(x) \) is translated right 4 units.

c. \( y = -2f(x) \)  
The graph is a vertical stretch of \( y = f(x) \) by a factor of 2 and reflected about the x-axis.

d. \( y = f(3x) \)  
The graph is a horizontal shrink of \( y = f(x) \) by a factor of \( \frac{1}{3} \).

e. \( y = 2f(x) - 5 \)  
The graph is a vertical stretch of \( y = f(x) \) by a factor of 2 and translated down 5 units.

3. The graph of the equation \( y = f(x) \) is provided below. For each of the following transformations of the graph, write a formula (in terms of \( f \)) for the function that is represented by the transformation of the graph of \( y = f(x) \). Then, draw the transformed graph of the function on the same set of axes as the graph of \( y = f(x) \).

a. A translation 3 units left and 2 units up  
\( p(x) = f(x + 3) + 2 \)
b. A vertical stretch by a scale factor of 3.
\[ q(x) = 3f(x) \]

c. A horizontal shrink by a scale factor of \( \frac{1}{2} \)
\[ r(x) = f(2x) \]

4. Reexamine your work on Exploratory Challenge 2 and Exercises 3 and 4 from this lesson. Parts (b) and (c) of Exploratory Challenge 2 asked how the equations \( y = 2f(x) \) and \( y = f(2x) \) could be graphed with the help of the strategic points found in part (a). In this problem, we investigate whether it is possible to determine the graphs of \( y = 2f(x) \) and \( y = f(2x) \) by working with the piecewise linear function \( f \) directly.

a. Write the function \( f \) in Exploratory Challenge 2 as a piecewise linear function.
\[ f(x) = \begin{cases} 
0.5x + 1.5, & -5 \leq x \leq -1 \\
1, & -1 < x < 3 \\
x - 2, & 3 \leq x \leq 5
\end{cases} \]

b. Let \( g(x) = 2f(x) \). Use the graph you sketched in Exploratory Challenge 2, part (b) of \( y = 2f(x) \) to write the formula for the function \( g \) as a piecewise linear function.
\[ g(x) = \begin{cases} 
x + 3, & -5 \leq x \leq -1 \\
2, & -1 < x < 3 \\
2x - 4, & 3 \leq x \leq 5
\end{cases} \]

c. Let \( h(x) = f(2x) \). Use the graph you sketched in Exploratory Challenge 2, part (c) of \( y = f(2x) \) to write the formula for the function \( h \) as a piecewise linear function.
\[ h(x) = \begin{cases} 
x + 1.5, & -2.5 \leq x \leq -0.5 \\
1, & -0.5 < x < 1.5 \\
2x - 2, & 1.5 \leq x \leq 2.5
\end{cases} \]
d. Compare the piecewise linear functions $g$ and $h$ to the piecewise linear function $f$. Did the expressions defining each piece change? If so, how? Did the domains of each piece change? If so how?

Function $g$: Each piece of the formula for $g$ is 2 times the corresponding piece of the formula for $f$. The domains are the same.

Function $h$: Each piece of the formula for $h$ is found by substituting $2x$ in for $x$ in the corresponding piece of the formula for $f$. The length of each interval in the domain of $h$ is $\frac{1}{2}$ the length of the corresponding interval in the domain of $f$. 