Lesson 22: Congruence Criteria for Triangles—SAS

Student Outcomes
- Students learn why any two triangles that satisfy the SAS congruence criterion must be congruent.

Lesson Notes
Lesson 22 begins to investigate criteria, or the indicators, of triangle congruence. Students are introduced to the concept in Grade 8 but justified the criteria of triangle congruence (i.e., ASA, SAS, and SSS) in a more hands-on manner, manipulating physical forms of triangles through rigid motions to determine whether or not a pair of triangles is congruent. In this lesson, students formally prove the triangle congruency criteria.

Note that in the exercises that follow, proofs may employ both statements of equality of measure of angles and lengths of segments and statements of congruence of angles and segments. While not introduced formally, it is intuitively clear that two segments are congruent if and only if they are equal in length; similarly, two angles are equal in measure if and only if they are congruent. That is, a segment can be mapped onto another if and only if they are equal in length, and an angle can be mapped onto another if and only if they are equal in measure. Another implication is that some of our key facts and discoveries may also be stated in terms of congruence, such as “Vertical angles are congruent.” Or, “If two lines are cut by a transversal such that a pair of alternate interior angles are congruent, then the lines are parallel.” Discuss these results with your students. Exercise 4 of this lesson should help students understand the logical equivalency of these statements.

Classwork
Opening Exercise (4 minutes)

Opening Exercise
Answer the following question. Then discuss your answer with a partner.

Do you think it is possible to know whether there is a rigid motion that takes one triangle to another without actually showing the particular rigid motion? Why or why not?

Answers may vary. Some students may think it is not possible because it is necessary to show the transformation as proof of its existence. Others may think it is possible by examining the triangles carefully.

It is common for curricula to take indicators of triangle congruence such as SAS and ASA as axiomatic, but this curriculum defines congruence in terms of rigid motions (as defined in the G-CO domain). However, it can be shown that these commonly used statements (SAS, ASA, etc.) follow from this definition of congruence and the properties of basic rigid motions (G-CO.B.8). Thus, these statements are indicators of whether rigid motions exist to take one triangle to the other. In other words, we have agreed to use the word congruent to mean there exists a composition of basic rigid motion of the plane that maps one figure to the other. We see that SAS, ASA, and SSS imply the existence of the rigid motion needed, but precision demands that we explain how and why.
While there are multiple proofs that show that SAS follows from the definition of congruence in terms of rigid motions and the properties of basic rigid motions, the one that appears in this lesson is one of the versions most accessible for students.

**Discussion (20 minutes)**

**Discussion**

It is true that we do not need to show the rigid motion to be able to know that there is one. We are going to show that there are criteria that refer to a few parts of the two triangles and a correspondence between them that guarantee congruency (i.e., existence of rigid motion). We start with the Side-Angle-Side (SAS) criteria.

**Side-Angle-Side Triangle Congruence Criteria (SAS):** Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ so that $AB = A'B'$ (Side), $m \angle A = m \angle A'$ (Angle), and $AC = A'C'$ (Side). Then the triangles are congruent.

The steps below show the most general case for determining a congruence between two triangles that satisfy the SAS criteria. Note that not all steps are needed for every pair of triangles. For example, sometimes the triangles already share a vertex. Sometimes a reflection is needed, sometimes not. It is important to understand that we can always use some or all of the steps below to determine a congruence between the two triangles that satisfies the SAS criteria.

**Proof:** Provided the two distinct triangles below, assume $AB = A'B'$ (Side), $m \angle A = m \angle A'$ (Angle), and $AC = A'C'$ (Side).

By our definition of congruence, we have to find a composition of rigid motions that maps $\triangle A'B'C'$ to $\triangle ABC$. We must find a congruence $F$ so that $(\triangle A'B'C') = \triangle ABC$. First, use a translation $T$ to map a common vertex.

Which two points determine the appropriate vector?

$A'$, $A$

Can any other pair of points be used? Why or why not?

No. We use $A'$ and $A$ because only these angles are congruent by assumption.
State the vector in the picture below that can be used to translate Δ A′B′C′.

\[ \overrightarrow{AA} \]

Using a dotted line, draw an intermediate position of Δ A′B′C′ as it moves along the vector:

After the translation (below), \( T_{\overrightarrow{AA}}(\Delta A'B'C') \) shares one vertex with \( \Delta ABC, A \). In fact, we can say

\[ T_{\overrightarrow{AA}}(\Delta A'B'C') = \Delta AB''C'' \]

Next, use a clockwise rotation \( R_{\angle CAA} \) to bring the side \( \overline{AB}'' \to \overline{AC} \) (or a counterclockwise rotation to bring \( \overline{AB}'' \) to \( \overline{AB} \)).

A rotation of appropriate measure maps \( \overrightarrow{AC''} \) to \( \overrightarrow{AC} \), but how can we be sure that vertex \( C'' \) maps to \( C \)? Recall that part of our assumption is that the lengths of sides in question are equal, ensuring that the rotation maps \( \overrightarrow{C''} \) to \( C \).

\( AC = AC'' \); the translation performed is a rigid motion, and thereby did not alter the length when \( A\overline{C} \) became \( A\overline{C''} \).

After the rotation \( R_{\angle CAC''}(\Delta AB''C'') \), a total of two vertices are shared with \( \Delta ABC, A \) and \( C \). Therefore,

\[ R_{\angle CAC''}(\Delta AB''C'') = \Delta AB'''C. \]
Finally, if $B''''$ and $B$ are on opposite sides of the line that joins $AC$, a reflection $r_{AC}$ brings $B''''$ to the same side as $B$.

Since a reflection is a rigid motion and it preserves angle measures, we know that $m\angle B''''AC = m\angle BAC$ and so $AB''''$ maps to $AB$. If, however, $AB''''$ coincides with $AB$, can we be certain that $B''''$ actually maps to $B$? We can, because not only are we certain that the rays coincide but also by our assumption that $AB = AB''''$. (Our assumption began as $AB = A'B'$, but the translation and rotation have preserved this length now as $AB''''$.)

Taken together, these two pieces of information ensure that the reflection over $AC$ brings $B''''$ to $B$.

Another way to visually confirm this is to draw the marks of the __perpendicular bisector__ construction for $AC$.

Write the transformations used to correctly notate the congruence (the composition of transformations) that take \( \triangle A'B'C' \cong \triangle ABC \):

\[
F \quad \text{Translation} \\
G \quad \text{Rotation} \\
H \quad \text{Reflection}
\]

\[
H \left( G \left( F \left( \triangle A'B'C' \right) \right) \right) = \triangle ABC
\]

We have now shown a sequence of rigid motions that takes $\triangle A'B'C'$ to $\triangle ABC$ with the use of just three criteria from each triangle: two sides and an included angle. Given any two distinct triangles, we could perform a similar proof. There is another situation when the triangles are not distinct, where a modified proof is needed to show that the triangles map onto each other. Examine these below. Note that when using the Side-Angle-Side triangle congruence criteria as a reason in a proof, you need only state the congruence and SAS.

**Example (5 minutes)**

Students try an example based on the Discussion.

**Example**

What if we had the SAS criteria for two triangles that were not distinct? Consider the following two cases. How would the transformations needed to demonstrate congruence change?

<table>
<thead>
<tr>
<th>Case</th>
<th>Diagram</th>
<th>Transformations Needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shared Side</td>
<td><img src="image" alt="Shared Side Diagram" /></td>
<td>reflection</td>
</tr>
<tr>
<td>Shared Vertex</td>
<td><img src="image" alt="Shared Vertex Diagram" /></td>
<td>rotation, reflection</td>
</tr>
</tbody>
</table>
Exercises (7 minutes)

1. Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure.
   Sketch and label three phases of the sequence of rigid motions that prove the two triangles to be congruent.

<table>
<thead>
<tr>
<th>Translation</th>
<th>Rotation</th>
<th>Reflection</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Translation" /></td>
<td><img src="image2" alt="Rotation" /></td>
<td><img src="image3" alt="Reflection" /></td>
</tr>
</tbody>
</table>

   Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

2. Given: $\angle LNM = \angle LNO$, $MN = ON$
   Do $\triangle LNM$ and $\triangle LNO$ meet the SAS criteria?

   $\angle LNM = \angle LNO$  \hspace{1cm} \text{Given}
   $MN = ON$  \hspace{1cm} \text{Given}
   $LN = LN$  \hspace{1cm} \text{Reflexive property}
   $\triangle LNM \cong \triangle LNO$  \hspace{1cm} \text{SAS}

   The triangles map onto one another with a reflection over $LN$.

3. Given: $\angle HGI = \angle JIG$, $HG = JI$
   Do $\triangle HGI$ and $\triangle JIG$ meet the SAS criteria?

   $\angle HGI = \angle JIG$  \hspace{1cm} \text{Given}
   $HG = JI$  \hspace{1cm} \text{Given}
   $GI = IG$  \hspace{1cm} \text{Reflexive property}
   $\triangle HGI \cong \triangle JIG$  \hspace{1cm} \text{SAS}

   The triangles map onto one another with a $180^\circ$ rotation about the midpoint of the diagonal.
4. Is it true that we could also have proved \( \triangle HGI \) and \( \triangle JIG \) meet the SAS criteria if we had been given that \( \angle HGI \cong \angle JIG \) and \( HG \equiv JI \)? Explain why or why not.

Yes, this is true. Whenever angles are equal, they can also be described as congruent. Since rigid motions preserve angle measure, for two angles of equal measure, there always exists a sequence of rigid motions that will carry one onto the other. Additionally, since rigid motions preserve distance, for two segments of equal length, there always exists a sequence of rigid motions that carries one onto the other.

Closing (1 minute)

- Two triangles, \( \triangle ABC \) and \( \triangle A'B'C' \), meet the Side-Angle-Side criteria when \( AB = A'B' \) (Side), \( m \angle A = m \angle A' \) (Angle), and \( AC = A'C' \) (Side). The SAS criteria implies the existence of a congruence that maps one triangle onto the other.

Exit Ticket (8 minutes)
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Exit Ticket

If two triangles satisfy the SAS criteria, describe the rigid motion(s) that would map one onto the other in the following cases.

1. The two triangles share a single common vertex.

2. The two triangles are distinct from each other.

3. The two triangles share a common side.
Exit Ticket Sample Solutions

If two triangles satisfy the SAS criteria, describe the rigid motion(s) that would map one onto the other in the following cases.

1. The two triangles share a single common vertex.
   Rotation, reflection

2. The two triangles are distinct from each other.
   Translation, rotation, reflection

3. The two triangles share a common side.
   Reflection

Problem Set Sample Solutions

Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

1. Given: \( \overline{AB} \parallel \overline{CD}, \) and \( AB = CD \)
   Do \( \triangle ABD \) and \( \triangle CDB \) meet the SAS criteria?
   
   \( AB = CD, \overline{AB} \parallel \overline{CD} \)  \hspace{1cm} Given
   \( BD = DB \)  \hspace{1cm} Reflexive property
   \( m\angle ABC = m\angle CDB \)  \hspace{1cm} If a transversal intersects two parallel lines, then the measures of the alternate interior angles are equal in measure.
   
   \( \triangle ABD \cong \triangle CDB \)  \hspace{1cm} SAS
   
   The triangles map onto one another with a 180° rotation about the midpoint of the diagonal.

2. Given: \( m\angle R = 25^\circ, RT = 7^\prime, SU = 5^\prime, \) and \( ST = 5^\prime \)
   Do \( \triangle RSU \) and \( \triangle RST \) meet the SAS criteria?
   
   There is not enough information given.

3. Given: \( \overline{KM} \) and \( \overline{JN} \) bisect each other
   Do \( \triangle JKL \) and \( \triangle NML \) meet the SAS criteria?
   
   \( \overline{KM} \) and \( \overline{JN} \) bisect each other  \hspace{1cm} Given
   \( m\angle KLF = m\angle MLP \)  \hspace{1cm} Vertical angles are equal in measure
   \( KL = ML \)  \hspace{1cm} Definition of a segment bisector
   \( JL = NL \)  \hspace{1cm} Definition of a segment bisector
   \( \triangle JKL \cong \triangle NML \)  \hspace{1cm} SAS
   
   The triangles map onto one another with a 180° rotation about \( L \).
4. Given: \( m \angle 1 = m \angle 2 \), and \( BC = DC \)
   Do \( \triangle ABC \) and \( \triangle ADC \) meet the SAS criteria?
   \[ m \angle 1 = m \angle 2 \quad \text{Given} \]
   \[ BC = DC \quad \text{Given} \]
   \[ AC = AC \quad \text{Reflexive property} \]
   \[ \triangle ABC \cong \triangle ADC \quad \text{SAS} \]
   The triangles map onto one another with a reflection over \( \overrightarrow{AC} \).

5. Given: \( \overline{AE} \) bisects angle \( \angle BCD \), and \( BC = DC \)
   Do \( \triangle CAB \) and \( \triangle CAD \) meet the SAS criteria?
   \[ \overline{AE} \text{ bisects angle } \angle BCD \quad \text{Given} \]
   \[ m \angle BCA = m \angle DCA \quad \text{Definition of an angle bisector} \]
   \[ BC = DC \quad \text{Given} \]
   \[ AC = AC \quad \text{Reflexive property} \]
   \[ \triangle CAD \cong \triangle CAB \quad \text{SAS} \]
   The triangles map onto one another with a reflection over \( \overrightarrow{AC} \).

6. Given: \( SU \) and \( RT \) bisect each other
   Do \( \triangle SVR \) and \( \triangle UVT \) meet the SAS criteria?
   \[ SU \text{ and } RT \text{ bisect each other} \quad \text{Given} \]
   \[ SV = UV \quad \text{Definition of a segment bisector} \]
   \[ RV = VT \quad \text{Definition of a segment bisector} \]
   \[ m \angle SVR = m \angle UVT \quad \text{Vertical angles are equal in measure} \]
   \[ \triangle SVR \cong \triangle UVT \quad \text{SAS} \]
   The triangles map onto one another with a 180° rotation about \( V \).

7. Given: \( JM = KL \), \( JM \perp ML \), and \( KL \perp ML \)
   Do \( \triangle JML \) and \( \triangle KLM \) meet the SAS criteria?
   \[ JM = KL \quad \text{Given} \]
   \[ JM \perp ML, KM \perp ML \quad \text{Given} \]
   \[ m \angle JML = 90^\circ, m \angle LKM = 90^\circ \quad \text{Definition of perpendicular lines} \]
   \[ m \angle JML = m \angle KLM \quad \text{Transitive property} \]
   \[ ML = LM \quad \text{Reflexive property} \]
   \[ \triangle JML \cong \triangle KLM \quad \text{SAS} \]
   The triangles map onto one another with a reflection over the perpendicular bisector of \( ML \).

8. Given: \( BF \perp AC \), and \( CE \perp AB \)
   Do \( \triangle BED \) and \( \triangle CFD \) meet the SAS criteria?
   There is not enough information given.
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9. Given: \( m\angle XY = m\angle VYX \)
   Do \( \triangle VXW \) and \( \triangle VYZ \) meet the SAS criteria?

   There is not enough information given.

10. Given: \( \triangle RST \) is isosceles, with \( RS = RT \), and \( SY = TZ \)
    Do \( \triangle RSY \) and \( \triangle RTZ \) meet the SAS criteria?

    \( \triangle RST \) is isosceles with \( RS = RT \)  
    \( m\angle S = m\angle T \)  
    \( SY = TZ \)  
    \( \triangle RSY \equiv \triangle RTZ \)  
    Given  
    Base angles of an isosceles triangle are equal in measure.  
    Given  
    SAS