Lesson 8: Adding and Subtracting Polynomials

Student Outcomes
- Students understand that the sum or difference of two polynomials produces another polynomial and relate polynomials to the system of integers; students add and subtract polynomials.

Classwork

Exercise 1 (7 minutes)
Have students complete Exercise 1 part (a) and use it for a brief discussion on the notion of base. Then have students continue with the remainder of the exercise.

Exercise 1

a. How many quarters, nickels, and pennies are needed to make $1.13?
   Answers will vary.
   4 quarters, 2 nickels, 3 pennies

b. Fill in the blanks:
   \[8,943 = \underline{8} \times 1000 + \underline{9} \times 100 + \underline{4} \times 10 + \underline{3} \times 1\]
   \[= \underline{8} \times 10^3 + \underline{9} \times 10^2 + \underline{4} \times 10 + \underline{3} \times 1\]

c. Fill in the blanks:
   \[8,943 = \underline{1} \times 20^3 + \underline{2} \times 20^2 + \underline{7} \times 20 + \underline{3} \times 1\]

d. Fill in the blanks:
   \[113 = \underline{4} \times 5^2 + \underline{2} \times 5 + \underline{3} \times 1\]

Next ask:
- Why do we use base 10? Why do we humans have a predilection for the number 10?
- Why do some cultures have base 20?
- How do you say 87 in French? How does the Gettysburg Address begin?
  - Quatre-vingt-sept: 4-20s and 7; Four score and seven years ago...
- Computers use which base system?
  - Base 2

Scaffolding:
- Mayan, Aztec, and Celtic cultures all used base 20. The word score (which means 20) originated from the Celtic language.
- Students could be asked to research more on this and on the cultures who use or used base 5 and base 60.
Exercise 2 (5 minutes)

In Exercise 2, we are laying the foundation that polynomials written in standard form are simply base $x$ numbers. The practice of filling in specific values for $x$ and finding the resulting values lays a foundation for connecting this algebra of polynomial expressions with the later lessons on polynomial functions (and other functions) and their inputs and outputs.

Work through Exercise 2 with the class.

<table>
<thead>
<tr>
<th>Exercise 2</th>
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<tbody>
<tr>
<td>Now let’s be as general as possible by not identifying which base we are in. Just call the base $x$.</td>
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<tr>
<td>Consider the expression $1 \cdot x^3 + 2 \cdot x^2 + 7 \cdot x + 3 \cdot 1$, or equivalently $x^3 + 2x^2 + 7x + 3$.</td>
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<tr>
<td>a. What is the value of this expression if $x = 10$?</td>
</tr>
<tr>
<td>1,273</td>
</tr>
<tr>
<td>b. What is the value of this expression if $x = 20$?</td>
</tr>
<tr>
<td>8,943</td>
</tr>
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Point out that the expression we see here is just the generalized form of their answer from part (b) of Exercise 1. However, as we change $x$, we get a different number each time.

Exercise 3 (10 minutes)

Allow students time to complete Exercise 3 individually. Then elicit responses from the class.

<table>
<thead>
<tr>
<th>Exercise 3</th>
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<tbody>
<tr>
<td>a. When writing numbers in base 10, we only allow coefficients of 0 through 9. Why is that?</td>
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<td><em>Once you get ten of a given unit, you also have one of the unit to the left of that.</em></td>
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<tr>
<td>b. What is the value of $22x + 3$ when $x = 5$? How much money is 22 nickels and 3 pennies?</td>
</tr>
<tr>
<td>113</td>
</tr>
<tr>
<td>$1.13$</td>
</tr>
<tr>
<td>c. What number is represented by $4x^2 + 17x + 2$ if $x = 10$?</td>
</tr>
<tr>
<td>572</td>
</tr>
<tr>
<td>d. What number is represented by $4x^2 + 17x + 2$ if $x = -2$ or if $x = \frac{2}{3}$?</td>
</tr>
<tr>
<td>$-16$</td>
</tr>
<tr>
<td>$136$</td>
</tr>
<tr>
<td>$9$</td>
</tr>
</tbody>
</table>
e. What number is represented by \(-3x^2 + \sqrt{2}x + \frac{1}{2}\) when \(x = \sqrt{2}\)?

\[-\frac{7}{2}\]

Point out, as highlighted by Exercises 1 and 3, that carrying is not necessary in this type of expression (polynomial expressions). For example, \(4x^2 + 17x + 2\) is a valid expression. However, in base ten arithmetic, coefficients of value ten or greater are not conventional notation. Setting \(x = 10\) in \(4x^2 + 17x + 2\) yields 4 hundreds, 17 tens, and 2 ones, which is to be expressed as 5 hundreds, 7 tens, and 2 ones.

Discussion (11 minutes)

- The next item in your student materials is a definition for a polynomial expression. Read the definition carefully, and then create 3 polynomial expressions using the given definition.

**POLYNOMIAL EXPRESSION:** A polynomial expression is either

1. A numerical expression or a variable symbol, or
2. The result of placing two previously generated polynomial expressions into the blanks of the addition operator \((\_+\_\)) or the multiplication operator \((\_\times\_\)).

- Compare your polynomial expressions with a neighbor’s. Do your neighbor’s expressions fall into the category of polynomial expressions?

Resolve any debates as to whether a given expression is indeed a polynomial expression by referring back to the definition and discussing as a class.

- Note that the definition of a polynomial expression includes subtraction (add the additive inverse instead), dividing by a nonzero number (multiply by the multiplicative inverse instead), and even exponentiation by a nonnegative integer (use the multiplication operator repeatedly on the same numerical or variable symbol).

List several of the student-generated polynomials on the board. Include some that contain more than one variable.

Initiate the following discussion, presenting expressions on the board when relevant.

- Just as the expression \((3 + 4) \cdot 5\) is a numerical expression but not a number, \((x + 5) + (2x^2 - x)(3x + 1)\) is a polynomial expression but not technically a polynomial. We reserve the word polynomial for polynomial expressions that are written simply as a sum of monomial terms. This begs the question: What is a monomial?

- A monomial is a polynomial expression generated using only the multiplication operator \((\_\times\_\)). Thus, it does not contain \(+\) or \(-\) operators.

- Just as we would not typically write a number in factored form and still refer to it as a number (we might call it a number in factored form), similarly, we do not write a monomial in factored form and still refer to it as a monomial. We multiply any numerical factors together and condense multiple instances of a variable factor using (whole number) exponents.

- Try creating a monomial.

- Compare the monomial you created with your neighbor’s. Is your neighbor’s expression really a monomial? Is it written in the standard form we use for monomials?
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- There are also such things as binomials and trinomials. Can anyone make a conjecture about what a binomial is and what a trinomial is and how they are the same or different from a polynomial?

Students may conjecture that a binomial has two of something and that a trinomial three of something. Further, they might conjecture that a polynomial has many of something. Allow for discussion and then state the following:

- A binomial is the sum (or difference) of two monomials. A trinomial is the sum (or difference) of three monomials. A polynomial, as stated earlier, is the sum of one or more monomials.
- The degree of a monomial is the sum of the exponents of the variable symbols that appear in the monomial.
- The degree of a polynomial is the degree of the monomial term with the highest degree.
- While polynomials can contain multiple variable symbols, most of our work with polynomials will be with polynomials in one variable.
- What do polynomial expressions in one variable look like? Create a polynomial expression in one variable, and compare with your neighbor.

Post some of the student-generated polynomials in one variable on the board.
- Let’s relate polynomials to the work we did at the beginning of the lesson.
- Is this expression an integer in base 10? 10(100 + 22 − 2) + 3(10) + 8 − 2(2)
- Is the expression equivalent to the integer 1,234?
- How did we find out?
- We rewrote the first expression in our standard form, right?
- Polynomials in one variable have a standard form as well. Use your intuition of what standard form of a polynomial might be to write this polynomial expression as a polynomial in standard form: 2x(x² − 3x + 1) − (x³ + 2), and compare your result with your neighbor.
  - Students should arrive at the answer x³ − 6x² + 2x − 2.

Confirm that in standard form, we start with the highest degree monomial and continue in descending order.
- The leading term of a polynomial is the term of highest degree that would be written first if the polynomial is put into standard form. The leading coefficient is the coefficient of the leading term.
- What would you imagine we mean when we refer to the constant term of the polynomial?
  - A constant term is any term with no variables. To find the constant term of a polynomial, be sure you have combined any and all constant terms into one single numerical term, written last if the polynomial is put into standard form. Note that a polynomial does not have to have a constant term (or could be said to have a constant term of 0).

As an extension for advanced students, assign the task of writing a formal definition for standard form of a polynomial. The formal definition is provided below for your reference:

A polynomial expression with one variable symbol \( x \) is in standard form if it is expressed as \( a_n x^n + a_{n-1} x^{n-1} + a_1 x + a_0 \), where \( n \) is a non-negative integer, and \( a_0, a_1, a_2, \ldots, a_n \) are constant coefficients with \( a_n \neq 0 \). A polynomial expression in \( x \) that is in standard form is often called a polynomial in \( x \).
Exercise 4 (5 minutes)

Find each sum or difference by combining the parts that are alike.

a. \(417 + 231 = \underline{4}\text{ hundreds} + \underline{1}\text{ tens} + \underline{7}\text{ ones} + \underline{2}\text{ hundreds} + \underline{3}\text{ tens} + \underline{1}\text{ ones}\)
\[= \underline{6}\text{ hundreds} + \underline{4}\text{ tens} + \underline{8}\text{ ones}\]

b. \((4x^2 + x + 7) + (2x^2 + 3x + 1)\)
\[6x^2 + 4x + 8\]

c. \((3x^3 - x^2 + 8) - (x^3 + 5x^2 + 4x - 7)\)
\[2x^3 - 6x^2 - 4x + 15\]

d. \(3(x^3 + 8x) - 2(x^3 + 12)\)
\[x^3 + 24x - 24\]

e. \((5 - t - t^2) + (9t + t^2)\)
\[8t + 5\]

f. \((3p + 1) + 6(p - 8) - (p + 2)\)
\[8p - 49\]

Closing (3 minutes)

- How are polynomials analogous to integers?
  - While integers are in base 10, polynomials are in base \(x\).
- If two polynomials are added together, is the result sure to be another polynomial? The difference of two polynomials?
  - Students will likely reply, “yes,” based on the few examples and their intuition.
- Are you sure? Can you think of an example where adding or subtracting two polynomials does not result in a polynomial?
  - Students thinking about \(x^2 - x^2 = 0\) could suggest not. At this point, review the definition of a polynomial. Constant symbols are polynomials.
Lesson Summary

A monomial is a polynomial expression generated using only the multiplication operator \( \times \). Thus, it does not contain + or − operators. Monomials are written with numerical factors multiplied together and variable or other symbols each occurring one time (using exponents to condense multiple instances of the same variable).

A polynomial is the sum (or difference) of monomials.

The degree of a monomial is the sum of the exponents of the variable symbols that appear in the monomial.

The degree of a polynomial is the degree of the monomial term with the highest degree.

Exit Ticket (4 minutes)
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Exit Ticket

1. Must the sum of three polynomials again be a polynomial?

2. Find \((w^2 - w + 1) + (w^3 - 2w^2 + 99)\).
Exit Ticket Sample Solutions

1. Must the sum of three polynomials again be a polynomial?
   Yes.

2. Find \((w^2 - w + 1) + (w^3 - 2w^2 + 99)\).
   \(w^3 - w^2 - w + 100\)

Problem Set Sample Solutions

1. Celina says that each of the following expressions is actually a binomial in disguise:
   i. \(5abc - 2a^2 + 6abc\)
   ii. \(5x^3 - 2x^2 - 10x^4 + 3x^5 + 3x \cdot (-2)x^4\)
   iii. \((t + 2)^2 - 4t\)
   iv. \(5(a - 1) - 10(a - 1) + 100(a - 1)\)
   v. \((2\pi r - \pi r^2)r - (2\pi r - \pi r^2) \cdot 2r\)
   For example, she sees that the expression in (i) is algebraically equivalent to \(11abc - 2a^2\), which is indeed a binomial. (She is happy to write this as \(11abc + (-2)a^2\), if you prefer.)
   Is she right about the remaining four expressions?
   *She is right about the remaining four expressions. They all can be expressed as binomials.*

2. Janie writes a polynomial expression using only one variable, \(x\), with degree 3. Max writes a polynomial expression using only one variable, \(x\), with degree 7.

   a. What can you determine about the degree of the sum of Janie’s and Max’s polynomials?
   *The degree would be 7.*

   b. What can you determine about the degree of the difference of Janie’s and Max’s polynomials?
   *The degree would be 7.*

3. Suppose Janie writes a polynomial expression using only one variable, \(x\), with degree of 5, and Max writes a polynomial expression using only one variable, \(x\), with degree of 5.

   a. What can you determine about the degree of the sum of Janie’s and Max’s polynomials?
   *The maximum degree could be 5, but it could also be anything less than that. For example, if Janie’s polynomial were \(x^5 + 3x - 1\), and Max’s were \(-x^5 + 2x^2 + 1\), the degree of the sum is only 2.*

   b. What can you determine about the degree of the difference of Janie’s and Max’s polynomials?
   *The maximum degree could be 5, but it could also be anything less than that.*
4. Find each sum or difference by combining the parts that are alike.
   a. \((2p + 4) + 5(p - 1) - (p + 7)\)
   \[6p - 8\]
   b. \((7x^4 + 9x) - 2(x^4 + 13)\)
   \[5x^4 + 9x - 26\]
   c. \((6 - t - t^2) + (9t + t^4)\)
   \[8t + 6\]
   d. \((5 - t^2) + 6(t^2 - 8) - (t^2 + 12)\)
   \[4t^2 - 55\]
   e. \((8x^3 + 5x) - 3(x^3 + 2)\)
   \[5x^3 + 5x - 6\]
   f. \((12x + 1) + 2(x - 4) - (x - 15)\)
   \[13x + 8\]
   g. \((13x^2 + 5x) - 2(x^2 + 1)\)
   \[11x^2 + 5x - 2\]
   h. \((9 - t - t^2) - \frac{3}{2}(8t + 2t^2)\)
   \[-4t^2 - 13t + 9\]
   i. \((4m + 6) - 12(m - 3) + (m + 2)\)
   \[-7m + 44\]
   j. \((15x^4 + 10x) - 12(x^4 + 4x)\)
   \[3x^4 - 38x\]