Lesson 16: Converse of the Pythagorean Theorem

Student Outcomes

- Students explain a proof of the converse of the Pythagorean theorem.
- Students apply the theorem and its converse to solve problems.

Lesson Notes

Students had their first experience with the converse of the Pythagorean theorem in Module 3 Lesson 14. In that lesson, students learned the proof of the converse by contradiction. That is, students were asked to draw a triangle with sides $a$, $b$, $c$, where the angle between sides $a$ and $b$ is different from $90^\circ$. The proof using the Pythagorean theorem led students to an expression that was not possible; that is, two times a length was equal to zero. This contradiction meant that the angle between sides $a$ and $b$ was in fact $90^\circ$. In this lesson, students are given two triangles with base and height dimensions of $a$ and $b$. They are told that one of the triangles is a right triangle and has lengths that satisfy the Pythagorean theorem. Students must use computation and their understanding of the basic rigid motions to show that the triangle with an unmarked angle is also a right triangle. The proof is subtle, so it is important from the beginning that students understand the differences between the triangles used in the discussion of the proof of the converse.

Classwork

Discussion (20 minutes)

- So far you have seen three different proofs of the Pythagorean theorem: **Theorem**: If the lengths of the legs of a right triangle are $a$ and $b$, and the length of the hypotenuse is $c$, then $a^2 + b^2 = c^2$.

Provide students time to explain to a partner a proof of the Pythagorean theorem. Allow them to choose any one of the three proofs they have seen. Remind them of the proof from Module 2 that was based on congruent triangles, knowledge about angle sum of a triangle, and angles on a line. Also remind them of the proof from Module 3 that was based on their knowledge of similar triangles and corresponding sides being equal in ratio. Select students to share their proofs with the class. Encourage other students to critique the reasoning of the student providing the proof.

- What do you recall about the meaning of the word converse?

Consider pointing out the hypothesis and conclusion of the Pythagorean theorem and then asking students to describe the converse in those terms.

- The converse is when the hypothesis and conclusion of a theorem are reversed.

Scaffolding:

Provide students samples of converses (and note that converses are not always true):

- If a whole number has a final digit of zero, then it is divisible by 10. Converse: If it is divisible by 10, then its final digit is zero.
- If it is raining, I will study inside the house. Converse: If I study inside the house, it is raining.
You have also seen one proof of the converse:

- If the lengths of three sides of a triangle $a$, $b$, and $c$ satisfy $c^2 = a^2 + b^2$, then the triangle is a right triangle, and furthermore, the side of length $c$ is opposite the right angle.

The following is another proof of the converse. Assume we are given a triangle $ABC$ so that the sides $a$, $b$, and $c$ satisfy $c^2 = a^2 + b^2$. We want to show that $\angle ACB$ is a right angle. To do so, we construct a right triangle $A'B'C'$ with leg lengths of $a$ and $b$ and right angle $\angle A'C'B'$.

### Proof of the Converse of the Pythagorean Theorem

- What do we know or not know about each of these triangles?
  - In the first triangle, $ABC$, we know that $a^2 + b^2 = c^2$. We do not know if angle $C$ is a right angle.
  - In the second triangle, $A'B'C'$, we know that it is a right triangle.

- What conclusions can we draw from this?
  - By applying the Pythagorean theorem to $\triangle A'B'C'$, we get $|A'B'|^2 = a^2 + b^2$. Since we are given $c^2 = a^2 + b^2$, then by substitution, $|A'B'|^2 = c^2$, and then $|A'B'| = c$. Since $c$ is also $|AB|$, then $|A'B'| = |AB|$. That means that both triangles have sides $a$, $b$, and $c$ that are the exact same lengths.

- Recall that we would like to prove that $\angle ACB$ is a right angle, that it maps to $\angle A'C'B'$. If we can translate $\triangle ABC$ so that $A$ goes to $A'$, $B$ goes to $B'$, and $C$ goes to $C'$, it follows that all three angles in the triangle will match. In particular, that $\angle ACB$ maps to the right angle $\angle A'C'B'$, and so is a right angle, too.

- We can certainly perform a translation that takes $B$ to $B'$ and $C$ to $C'$ because segments $BC$ and $B'C'$ are the same length. Must this translation take $A$ to $A'$? What goes wrong mathematically if it misses and translates to a different point $A''$ as shown below?

In this picture, we’ve drawn $A''$ to the left of $A'C'$. The reasoning that follows works just as well for a picture with $A''$ to the right of $A'C'$ instead.
Provide time for students to think of what may go wrong mathematically. If needed, prompt them to notice the two isosceles triangles in the diagram, \( \triangle A''C'A' \) and \( \triangle A''B'A' \) and the four angles \( w_1, w_2, w_3, w_4 \) labeled as shown in the diagram below.

\[ \triangle A''C'A' \text{ is isosceles and therefore has base angles that are equal in measure:} \]
\[ w_1 + w_2 = w_3. \]

\[ \triangle A''B'A' \text{ is isosceles and therefore has base angles that are equal in measure:} \]
\[ w_2 = w_3 + w_4. \]

These two equations give \( w_1 + w_3 + w_4 = w_3, \) which is equal to \( w_1 + w_4 = 0, \) which is obviously not true.

- Therefore, the translation must map \( A \) to \( A', \) and since translations preserve the measures of angles, we can conclude that the measure of \( \angle ACB \) is equal to the measure of \( \angle A'C'B', \) and \( \angle ACB \) is a right angle.
- Finally, if a triangle has side lengths of \( a, b \) and \( c, \) with \( c \) the longest length, that don’t satisfy the equation \( a^2 + b^2 = c^2, \) then the triangle cannot be a right triangle.
Provide students time to explain to a partner a proof of the converse of the Pythagorean theorem. Allow them to choose either proof that they have seen. Remind them of the proof from Module 3 that was a proof by contradiction, where we assumed that the triangle was not a right triangle and then showed that the assumption was wrong. Select students to share their proofs with the class. Encourage other students to critique the reasoning of the student providing the proof.

Exercises 1–7 (15 minutes)

Students complete Exercises 1–7 independently. Remind students that since each of the exercises references the side length of a triangle, we need only consider the positive square root of each number because we cannot have a negative length.

**Exercises 1–7**

1. Is the triangle with leg lengths of 3 mi. and 8 mi. and hypotenuse of length $\sqrt{73}$ mi. a right triangle? Show your work, and answer in a complete sentence.

   $3^2 + 8^2 = (\sqrt{73})^2$
   $9 + 64 = 73$
   $73 = 73$

   **Yes, the triangle with leg lengths of 3 mi. and 8 mi. and hypotenuse of length $\sqrt{73}$ mi. is a right triangle because it satisfies the Pythagorean theorem.**

2. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

   Let $c$ in. represent the length of the hypotenuse of the triangle.

   $1^2 + 4^2 = c^2$
   $1 + 16 = c^2$
   $17 = c^2$
   $\sqrt{17} = c$
   $4.1 \approx c$

   **The length of the hypotenuse of the right triangle is exactly $\sqrt{17}$ inches and approximately 4.1 inches.**
3. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

![Right Triangle Diagram]

Let \( c \) mm represent the length of the hypotenuse of the triangle.

\[
2^2 + 6^2 = c^2 \\
4 + 36 = c^2 \\
40 = c^2 \\
\sqrt{40} = c \\
\sqrt{2^2 \times 5} = c \\
\sqrt{2^2} \times \sqrt{2} \times \sqrt{5} = c \\
2 \sqrt{10} = c
\]

The length of the hypotenuse of the right triangle is exactly \( 2\sqrt{10} \) mm and approximately 6.3 mm.

4. Is the triangle with leg lengths of 9 in. and 9 in. and hypotenuse of length \( \sqrt{175} \) in. a right triangle? Show your work, and answer in a complete sentence.

\[
9^2 + 9^2 = (\sqrt{175})^2 \\
81 + 81 = 175 \\
162 \neq 175
\]

No, the triangle with leg lengths of 9 in. and 9 in. and hypotenuse of length \( \sqrt{175} \) in. is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

5. Is the triangle with leg lengths of \( \sqrt{28} \) cm and 6 cm and hypotenuse of length 8 cm a right triangle? Show your work, and answer in a complete sentence.

\[
(\sqrt{28})^2 + 6^2 = 8^2 \\
28 + 36 = 64 \\
64 = 64
\]

Yes, the triangle with leg lengths of \( \sqrt{28} \) cm and 6 cm and hypotenuse of length 8 cm is a right triangle because the lengths satisfy the Pythagorean theorem.
6. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence.

Let \( c \) ft. represent the length of the hypotenuse of the triangle.

\[
3^2 + (\sqrt{27})^2 = c^2 \\
9 + 27 = c^2 \\
36 = c^2 \\
\sqrt{36} = \sqrt{c^2} \\
6 = c
\]

The length of the hypotenuse of the right triangle is 6 ft.

7. The triangle shown below is an isosceles right triangle. Determine the length of the legs of the triangle. Show your work, and answer in a complete sentence.

Let \( x \) cm represent the length of each of the legs of the isosceles triangle.

\[
x^2 + x^2 = (\sqrt{18})^2 \\
2x^2 = 18 \\
2x^2 = \frac{18}{2} \\
x^2 = 9 \\
\sqrt{x^2} = \sqrt{9} \\
x = 3
\]

The leg lengths of the isosceles triangle are 3 cm.

Closing (5 minutes)
Summarize, or ask students to summarize, the main points from the lesson.

- The converse of the Pythagorean theorem states that if side lengths of a triangle \( a, b, c \) satisfy \( a^2 + b^2 = c^2 \), then the triangle is a right triangle.
- If the side lengths of a triangle \( a, b, c \) do not satisfy \( a^2 + b^2 = c^2 \), then the triangle is not a right triangle.
- We know how to explain a proof of the Pythagorean theorem and its converse.

Lesson Summary
The converse of the Pythagorean theorem states that if a triangle with side lengths \( a, b, \) and \( c \) satisfies \( a^2 + b^2 = c^2 \), then the triangle is a right triangle.
The converse can be proven using concepts related to congruence.

Exit Ticket (5 minutes)
Lesson 16: Converse of the Pythagorean Theorem

Exit Ticket

1. Is the triangle with leg lengths of 7 mm and 7 mm and a hypotenuse of length 10 mm a right triangle? Show your work, and answer in a complete sentence.

2. What would the length of the hypotenuse need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.

3. If one of the leg lengths is 7 mm, what would the other leg length need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.
Exit Ticket Sample Solutions

1. **Is the triangle with leg lengths of 7 mm and 7 mm and a hypotenuse of length 10 mm a right triangle? Show your work, and answer in a complete sentence.**

   \[7^2 + 7^2 = 10^2\]
   \[49 + 49 = 100\]
   \[98 \neq 100\]

   *No, the triangle with leg lengths of 7 mm and 7 mm and hypotenuse of length 10 mm is not a right triangle because the lengths do not satisfy the Pythagorean theorem.*

2. **What would the length of the hypotenuse need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.**

   *Let \(c\) mm represent the length of the hypotenuse.*

   Then,

   \[7^2 + 7^2 = c^2\]
   \[49 + 49 = c^2\]
   \[98 = c^2\]
   \[\sqrt{98} = c\]

   *The hypotenuse would need to be \(\sqrt{98}\) mm for the triangle with sides of 7 mm and 7 mm to be a right triangle.*

3. **If one of the leg lengths is 7 mm, what would the other leg length need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.**

   *Let \(a\) mm represent the length of one leg.*

   Then,

   \[a^2 + 7^2 = 10^2\]
   \[a^2 + 49 = 100\]
   \[a^2 + 49 - 49 = 100 - 49\]
   \[a^2 = 51\]
   \[a = \sqrt{51}\]

   *The leg length would need to be \(\sqrt{51}\) mm so that the triangle with one leg length of 7 mm and the hypotenuse of 10 mm is a right triangle.*
Problem Set Sample Solutions

1. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

![Diagram of right triangle with sides 1 cm and 1 cm]

Let $c$ cm represent the length of the hypotenuse of the triangle.

$$1^2 + 1^2 = c^2$$
$$2 = c^2$$
$$\sqrt{2} = c$$

The length of the hypotenuse is exactly $\sqrt{2}$ cm and approximately 1.4 cm.

2. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

![Diagram of right triangle with sides 7 ft and 11 ft]

Let $x$ ft. represent the unknown length of the triangle.

$$7^2 + x^2 = 11^2$$
$$49 + x^2 = 121$$
$$49 - 49 + x^2 = 121 - 49$$
$$x^2 = 72$$
$$\sqrt{x^2} = \sqrt{72}$$
$$x = \sqrt{2 \cdot \sqrt{2} \cdot \sqrt{3}^2}$$
$$x = 6\sqrt{2}$$
$$x \approx 8.5$$

The length of the unknown side of the triangle is exactly $6\sqrt{2}$ ft. and approximately 8.5 ft.

3. Is the triangle with leg lengths of $\sqrt{3}$ cm and 9 cm and hypotenuse of length $\sqrt{84}$ cm a right triangle? Show your work, and answer in a complete sentence.

$$\left(\sqrt{3}\right)^2 + 9^2 = \left(\sqrt{84}\right)^2$$
$$3 + 81 = 84$$
$$84 = 84$$

Yes, the triangle with leg lengths of $\sqrt{3}$ cm and 9 cm and hypotenuse of length $\sqrt{84}$ cm is a right triangle because the lengths satisfy the Pythagorean theorem.

4. Is the triangle with leg lengths of $\sqrt{7}$ km and 5 km and hypotenuse of length $\sqrt{48}$ km a right triangle? Show your work, and answer in a complete sentence.

$$\left(\sqrt{7}\right)^2 + 5^2 = \left(\sqrt{48}\right)^2$$
$$7 + 25 = 48$$
$$32 \neq 48$$

No, the triangle with leg lengths of $\sqrt{7}$ km and 5 km and hypotenuse of length $\sqrt{48}$ km is not a right triangle because the lengths do not satisfy the Pythagorean theorem.
5. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

Let \( c \) mm represent the length of the hypotenuse of the triangle.

\[
5^2 + 10^2 = c^2 \\
25 + 100 = c^2 \\
125 = c^2 \\
\sqrt{125} = \sqrt{c^2} \\
\sqrt{5^2} = c \\
5 \sqrt{5} = c \\
11.2 \approx c
\]

The length of the hypotenuse is exactly \( 5 \sqrt{5} \) mm and approximately 11.2 mm.

6. Is the triangle with leg lengths of 3 and 6 and hypotenuse of length \( \sqrt{45} \) a right triangle? Show your work, and answer in a complete sentence.

\[
3^2 + 6^2 = (\sqrt{45})^2 \\
9 + 36 = 45 \\
45 = 45
\]

Yes, the triangle with leg lengths of 3 and 6 and hypotenuse of length \( \sqrt{45} \) is a right triangle because the lengths satisfy the Pythagorean theorem.

7. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

Let \( x \) in. represent the unknown side length of the triangle.

\[
2^2 + x^2 = 8^2 \\
4 + x^2 = 64 \\
4 - 4 + x^2 = 64 - 4 \\
x^2 = 60 \\
\sqrt{x^2} = \sqrt{60} \\
x = \sqrt{2^2 \cdot 3 \cdot 5} \\
x = 2\sqrt{15} \\
x \approx 7.7
\]

The length of the unknown side of the triangle is exactly \( 2\sqrt{15} \) inches and approximately 7.7 inches.

8. Is the triangle with leg lengths of 1 and \( \sqrt{3} \) and hypotenuse of length 2 a right triangle? Show your work, and answer in a complete sentence.

\[
1^2 + (\sqrt{3})^2 = 2^2 \\
1 + 3 = 4 \\
4 = 4
\]

Yes, the triangle with leg lengths of 1 and \( \sqrt{3} \) and hypotenuse of length 2 is a right triangle because the lengths satisfy the Pythagorean theorem.
9. Corey found the hypotenuse of a right triangle with leg lengths of 2 and 3 to be $\sqrt{13}$. Corey claims that since $\sqrt{13} \approx 3.61$ when estimating to two decimal digits, that a triangle with leg lengths of 2 and 3 and a hypotenuse of 3.61 is a right triangle. Is he correct? Explain.

No, Corey is not correct.

$2^2 + 3^2 = (3.61)^2$
$4 + 9 = 13.0321$
$13 \neq 13.0321$

No, the triangle with leg lengths of 2 and 3 and hypotenuse of length 3.61 is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

10. Explain a proof of the Pythagorean theorem.

Consider having students share their proof with a partner while their partner critiques their reasoning. Accept any of the three proofs that students have seen.

11. Explain a proof of the converse of the Pythagorean theorem.

Consider having students share their proof with a partner while their partner critiques their reasoning. Accept either of the proofs that students have seen.