Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology

Student Outcomes

- Students compare numbers expressed in scientific notation.
- Students apply the laws of exponents to interpret data and use technology to compute with very large numbers.

Classwork

Examples 1–2/ Exercises 1–2 (10 minutes)

Concept Development: We have learned why scientific notation is indispensable in science. This means that we have to learn how to compute and compare numbers in scientific notation. We have already done some computations, so we are ready to take a closer look at comparing the size of different numbers.

There is a general principle that underlies the comparison of two numbers in scientific notation: *Reduce everything to whole numbers if possible.* To this end, we recall two basic facts.

1. **Inequality (A): Let x and y be numbers and let $z > 0$. Then $x < y$ if and only if $xz < yz$.**
2. **Comparison of whole numbers:**
   a. **If two whole numbers have different numbers of digits, then the one with more digits is greater.**
   b. **Suppose two whole numbers p and q have the same number of digits and, moreover, they agree digit-by-digit (starting from the left) until the n^{th} place. If the digit of p in the $(n+1)^{\text{th}}$ place is greater than the corresponding digit in q, then $p > q$.**

Example 1

MP.8

Among the galaxies closest to Earth, M82 is about 1.15×10^7 light-years away, and Leo I Dwarf is about 8.2×10^5 light-years away. Which is closer?

- *First solution:* This is the down-to-earth, quick, and direct solution. The number 8.2×10^5 equals the 6-digit number $820{,}000$. On the other hand, 1.15×10^7 equals the 8-digit number $11{,}500{,}000$. By (2a), above, $8.2 \times 10^5 < 1.15 \times 10^7$. Therefore, Leo I Dwarf is closer.

MP.8

- *Second Solution:* This solution is for the long haul, that is, the solution that works every time no matter how large (or small) the numbers become. First, we express both numbers as a product with the same power of 10. Since $10^7 = 10^2 \times 10^5$, we see that the distance to M82 is

$$1.15 \times 10^2 \times 10^5 = 115 \times 10^5.$$

The distance to Leo I Dwarf is 8.2×10^5. By (1) above, comparing 1.15×10^7 and 8.2×10^5 is equivalent to comparing 115 and 8.2. Since $8.2 < 115$, we see that $8.2 \times 10^5 < 1.15 \times 10^7$. Thus, Leo I Dwarf is closer.

Scaffolding:

- Display the second solution.
- Guide students through the solution.

Exercise 1

Have students complete Exercise 1 independently, using the logic modeled in the second solution.

Exercise 1

The Fornax Dwarf galaxy is 4.6×10^5 light-years away from Earth, while Andromeda I is 2.430×10^6 light-years away from Earth. Which is closer to Earth?

$$2.430 \times 10^6 = 2.430 \times 10 \times 10^5 = 24.30 \times 10^5$$

Because $4.6 < 24.30$, then $4.6 \times 10^5 < 24.30 \times 10^5$, and since $24.30 \times 10^5 = 2.430 \times 10^6$, we know that $4.6 \times 10^5 < 2.430 \times 10^6$. Therefore, Fornax Dwarf is closer to Earth.

Example 2

Background information for the teacher: The next example brings us back to the world of subatomic particles. In the early 20th century, the picture of elementary particles was straightforward: Electrons, protons, neutrons, and photons were the fundamental constituents of matter. But in the 1930s, positrons, mesons, and neutrinos were discovered, and subsequent developments rapidly increased the number of subatomic particle types observed. Many of these newly observed particle types are extremely short-lived (see Example 2 below and Exercise 2). The so-called Standard Model developed during the latter part of the last century finally restored some order, and it is now theorized that different kinds of quarks and leptons are the basic constituents of matter.

Many subatomic particles are unstable: charged pions have an average lifetime of 2.603×10^{-8} seconds, while muons have an average lifetime of 2.197×10^{-6} seconds. Which has a longer average lifetime?

We follow the same method as the second solution in Example 1. We have

$$2.197 \times 10^{-6} = 2.197 \times 10^2 \times 10^{-8} = 219.7 \times 10^{-8}.$$

Therefore, comparing 2.603×10^{-8} with 2.197×10^{-6} is equivalent to comparing 2.603 with 219.7 (by (1) above). Since $2.603 < 219.7$, we see that $2.603 \times 10^{-8} < 2.197 \times 10^{-6}$. Thus, muons have a longer lifetime.

Exercise 2 (3 minutes)

Have students complete Exercise 2 independently.

Exercise 2

The average lifetime of the tau lepton is 2.906×10^{-13} seconds, and the average lifetime of the neutral pion is 8.4×10^{-17} seconds. Explain which subatomic particle has a longer average lifetime.

$$2.906 \times 10^{-13} = 2.906 \times 10^{4} \times 10^{-17} = 29{,}060 \times 10^{-17}$$

Since $8.4 < 29{,}060$, then $8.4 \times 10^{-17} < 29{,}060 \times 10^{-17}$, and since $29{,}060 \times 10^{-17} = 2.906 \times 10^{-13}$, we know that $8.4 \times 10^{-17} < 2.906 \times 10^{-13}$. Therefore, tau lepton has a longer average lifetime.

This problem, as well as others, can be solved using an alternate method. Our goal is to make the magnitude of the numbers we are comparing the same, which will allow us to reduce the comparison to that of whole numbers.

Here is an alternate solution:

$$8.4 \times 10^{-17} = 8.4 \times 10^{-4} \times 10^{-13} = 0.000\,84 \times 10^{-13}.$$

Since $0.00084 < 2.906$, then $0.000\,84 \times 10^{-13} < 2.906 \times 10^{-13}$, and since $0.000\,84 \times 10^{-13} = 8.4 \times 10^{-17}$, we know that $8.4 \times 10^{-17} < 2.906 \times 10^{-13}$. Therefore, tau lepton has a longer average lifetime.

Exploratory Challenge 1/Exercise 3 (8 minutes)

Examples 1 and 2 illustrate the following general fact:

MP.8

THEOREM: *Given two positive numbers in scientific notation, $a \times 10^m$* and *$b \times 10^n$, if $m < n$, then $a \times 10^m < b \times 10^n$.* Allow time for students to discuss, in small groups, how to prove the theorem.

Exploratory Challenge 1/Exercise 3

THEOREM: Given two positive numbers in scientific notation, $a \times 10^m$ and $b \times 10^n$, if $m < n$, then $a \times 10^m < b \times 10^n$.

Prove the theorem.

If $m < n$, then there is a positive integer k so that $n = k + m$.

By the first law of exponents (10) in Lesson 5, $b \times 10^n = b \times 10^k \times 10^m = (b \times 10^k) \times 10^m$. Because we are comparing with $a \times 10^m$, we know by (1) that we only need to prove $a < (b \times 10^k)$. By the definition of scientific notation, $a < 10$ and also $(b \times 10^k) \geq 10$ because $k \geq 1$ and $b \geq 1$, so that $(b \times 10^k) \geq 1 \times 10 = 10$. This proves $a < (b \times 10^k)$, and therefore, $a \times 10^m < b \times 10^n$.

Explain to students that we know that $a < 10$ because of the statement given that $a \times 10^m$ is a number expressed in scientific notation. That is not enough information to convince students that $a < b \times 10^k$; therefore, we need to say something about the right side of the inequality. We know that $k \geq 1$ because k is a positive integer so that $n = k + m$. We also know that $b \geq 1$ because of the definition of scientific notation. That means that the minimum possible value of $b \times 10^k$ is 10 because $1 \times 10^1 = 10$. Therefore, we can be certain that $a < b \times 10^k$.

Therefore, by (1), $a \times 10^m < (b \times 10^k) \times 10^m$. Since $n = k + m$, we can rewrite the right side of the inequality as $b \times 10^n$, and finally $a \times 10^m < b \times 10^n$.

Scaffolding:

- Use the suggestions below, as needed, for the work related to the theorem.
- Remind students about order of magnitude.
- Remind them that if $m < n$, then there is a positive integer k so that $n = k + m$. Therefore, by the first law of exponents (10),

$$b \times 10^n = b \times 10^k \times 10^m = (b \times 10^k) \times 10^m.$$

- Point out that we just spent time on forcing numbers that were expressed in scientific notation to have the same power of 10, which allowed us to easily compare the numbers. This proof is no different. We just wrote an equivalent expression $(b \times 10^k) \times 10^m$ for $b \times 10^n$, so that we could look at and compare two numbers that both have a magnitude of m.

Example 3 (2 minutes)

Compare 1.815×10^{14} with 1.82×10^{14}.

By (1), we only have to compare 1.815 with 1.82, and for the same reason, we only need to compare 1.815×10^3 with 1.82×10^3.

Thus, we compare $1{,}815$ and $1{,}820$: Clearly $1{,}815 < 1{,}820$ (use (2b) if you like). Therefore, using (1) repeatedly,

$$1{,}815 < 1{,}820 \rightarrow 1.815 < 1.82 \rightarrow 1.815 \times 10^{14} < 1.82 \times 10^{14}.$$

> *Scaffolding:*
>
> Remind students that it is easier to compare whole numbers; that's why each number is multiplied by 10^3. However, if students can accurately compare 1.815 to 1.82, it is not necessary that they multiply each number by 10^3 to make them whole numbers.

Exercises 4–5 (2 minutes)

Have students complete Exercises 4 and 5 independently.

Exercise 4

Compare 9.3×10^{28} and 9.2879×10^{28}.

We only need to compare 9.3 and 9.2879. $9.3 \times 10^4 = 93,000$ and $9.2879 \times 10^4 = 92,879$, so we see that $93,000 > 92,879$. Therefore, $9.3 \times 10^{28} > 9.2879 \times 10^{28}$.

Exercise 5

Chris said that $5.3 \times 10^{41} < 5.301 \times 10^{41}$ because 5.3 has fewer digits than 5.301. Show that even though his answer is correct, his reasoning is flawed. Show him an example to illustrate that his reasoning would result in an incorrect answer. Explain.

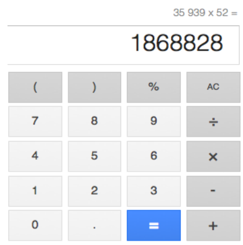
Chris is correct that $5.3 \times 10^{41} < 5.301 \times 10^{41}$, but that is because when we compare 5.3 and 5.301, we only need to compare 5.3×10^3 and 5.301×10^3 (by (1) above). But, $5.3 \times 10^3 < 5.301 \times 10^3$ or rather $5,300 < 5,301$, and this is the reason that $5.3 \times 10^{41} < 5.301 \times 10^{41}$. However, Chris's reasoning would lead to an incorrect answer for a problem that compares 5.9×10^{41} and 5.199×10^{41}. His reasoning would lead him to conclude that $5.9 \times 10^{41} < 5.199 \times 10^{41}$, but $5,900 > 5,199$, which is equivalent to $5.9 \times 10^3 > 5.199 \times 10^3$. By (1) again, $5.9 > 5.199$, meaning that $5.9 \times 10^{41} > 5.199 \times 10^{41}$.

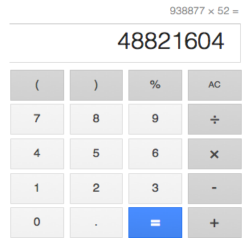
Exploratory Challenge 2/Exercise 6 (10 minutes)

Students use snapshots of technology displays to determine the exact product of two numbers.

Exploratory Challenge 2/Exercise 6

You have been asked to determine the exact number of Google searches that are made each year. The only information you are provided is that there are $35,939,938,877$ searches performed each week. Assuming the exact same number of searches are performed each week for the 52 weeks in a year, how many total searches will have been performed in one year? Your calculator does not display enough digits to get the exact answer. Therefore, you must break down the problem into smaller parts. Remember, you cannot approximate an answer because you need to find an exact answer. Use the screen shots below to help you reach your answer.





First, I need to rewrite the number of searches for each week using numbers that can be computed using my calculator.

$$\begin{aligned} 35\,939\,938\,877 &= 35\,939\,000\,000 + 938\,877 \\ &= 35\,939 \times 10^6 + 938\,877 \end{aligned}$$

Next, I need to multiply each term of the sum by 52, using the distributive law.

$$(35\,939 \times 10^6 + 938\,877) \times 52 = (35\,939 \times 10^6) \times 52 + (938\,877 \times 52)$$

By repeated use of the commutative and associative properties, I can rewrite the problem as

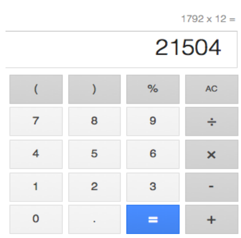
$$(35\,939 \times 52) \times 10^6 + (938\,877 \times 52).$$

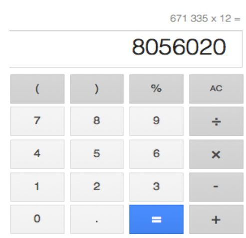
According to the screen shots, I get

$$\begin{aligned} 1\,868\,828 \times 10^6 + 48\,821\,604 &= 1\,868\,828\,000\,000 + 48\,821\,604 \\ &= 1\,868\,876\,821\,604. \end{aligned}$$

Therefore, $1,868,876,821,604$ Google searches are performed each year.

Yahoo! is another popular search engine. Yahoo! receives requests for $1,792,671,335$ searches each month. Assuming the same number of searches are performed each month, how many searches are performed on Yahoo! each year? Use the screen shots below to help determine the answer.





First, I need to rewrite the number of searches for each month using numbers that can be computed using my calculator.

$$\begin{aligned} 1\,792\,671\,335 &= 1\,792\,000\,000 + 671\,335 \\ &= 1\,792 \times 10^6 + 671\,335. \end{aligned}$$

Next, I need to multiply each term of the sum by 12, using the distributive law.

$$(1\,792 \times 10^6 + 671\,335) \times 12 = (1\,792 \times 10^6) \times 12 + (671\,335 \times 12).$$

By repeated use of the commutative and associative properties, I can rewrite the problem as

$$(1\,792 \times 12) \times 10^6 + (671\,335 \times 12)$$

According to the screen shots, I get

$$21\,504 \times 10^6 + 8\,056\,020 = 21\,504\,000\,000 + 8\,056\,020$$

$$= 21\,512\,056\,020$$

Therefore, $21,512,056,020$ Yahoo! searches are performed each year.

Closing (2 minutes)

Summarize the lesson and Module 1:

- We have completed the lessons on exponential notation, the properties of integer exponents, magnitude, and scientific notation.
- We can read, write, and operate with numbers expressed in scientific notation, which is the language of many sciences. Additionally, they can interpret data using technology.

Exit Ticket (3 minutes)

Fluency Exercise (5 minutes)

Rapid White Board Exchange: Have students respond to your prompts for practice with operations with numbers expressed in scientific notation using white boards (or other display options as available). This exercise can be conducted at any point throughout the lesson. The prompts are listed at the end of the lesson. Refer to the Rapid White Board Exchanges section in the Module Overview for directions to administer a Rapid White Board Exchange.

Name ____ Date ____________________

Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology

Exit Ticket

1. Compare 2.01×10^{15} and 2.8×10^{13}. Which number is larger?

2. The wavelength of the color red is about 6.5×10^{-9} m. The wavelength of the color blue is about 4.75×10^{-9} m. Show that the wavelength of red is longer than the wavelength of blue.

Exit Ticket Sample Solutions

1. Compare 2.01×10^{15} and 2.8×10^{13}. Which number is larger?

   $2.01 \times 10^{15} = 2.01 \times 10^{2} \times 10^{13} = 201 \times 10^{13}$

   Since $201 > 2.8$, we have $201 \times 10^{13} > 2.8 \times 10^{13}$, and since $201 \times 10^{13} = 2.01 \times 10^{15}$, we conclude $2.01 \times 10^{15} > 2.8 \times 10^{13}$.

2. The wavelength of the color red is about 6.5×10^{-9} m. The wavelength of the color blue is about 4.75×10^{-9} m. Show that the wavelength of red is longer than the wavelength of blue.

   We only need to compare 6.5 and 4.75:

   $6.5 \times 10^{-9} = 650 \times 10^{-7}$ and $4.75 \times 10^{-9} = 475 \times 10^{-7}$, so we see that $650 > 475$. Therefore, $6.5 \times 10^{-9} > 4.75 \times 10^{-9}$.

Problem Set Sample Solutions

1. Write out a detailed proof of the fact that, given two numbers in scientific notation, $a \times 10^{n}$ and $b \times 10^{n}$, $a < b$, if and only if $a \times 10^{n} < b \times 10^{n}$.

   Because $10^{n} > 0$, we can use inequality (A) (i.e., (1) above) twice to draw the necessary conclusions. First, if $a < b$, then by inequality (A), $a \times 10^{n} < b \times 10^{n}$. Second, given $a \times 10^{n} < b \times 10^{n}$, we can use inequality (A) again to show $a < b$ by multiplying each side of $a \times 10^{n} < b \times 10^{n}$ by 10^{-n}.

   a. Let A and B be two positive numbers, with no restrictions on their size. Is it true that $A \times 10^{-5} < B \times 10^{5}$?

      No, it is not true that $A \times 10^{-5} < B \times 10^{5}$. Using inequality (A), we can write $A \times 10^{-5} \times 10^{5} < B \times 10^{5} \times 10^{5}$, which is the same as $A < B \times 10^{10}$. To disprove the statement, all we would need to do is find a value of A that exceeds $B \times 10^{10}$.

   b. Now, if $A \times 10^{-5}$ and $B \times 10^{5}$ are written in scientific notation, is it true that $A \times 10^{-5} < B \times 10^{5}$? Explain.

      Yes, since the numbers are written in scientific notation, we know that the restrictions for A and B are $1 \le A < 10$ and $1 \le B < 10$. The maximum value for A, when multiplied by 10^{-5}, will still be less than 1. The minimum value of B will produce a number at least 10^{5} in size.

2. The mass of a neutron is approximately 1.674927×10^{-27} kg. Recall that the mass of a proton is 1.672622×10^{-27} kg. Explain which is heavier.

   Since both numbers have a factor of 10^{-27}, we only need to look at 1.674927 and 1.672622. When we multiply each number by 10^{6}, we get

   $$1.674927 \times 10^{6} \text{ and } 1.672622 \times 10^{6},$$

   which is the same as

   $$1{,}674{,}927 \text{ and } 1{,}672{,}622.$$

   Now that we are looking at whole numbers, we can see that $1{,}674{,}927 > 1{,}672{,}622$ (by (2b) above), which means that $1.674927 \times 10^{-27} > 1.672622 \times 10^{-27}$. Therefore, the mass of a neutron is heavier.

3. The average lifetime of the Z boson is approximately 3×10^{-25} seconds, and the average lifetime of a neutral rho meson is approximately 4.5×10^{-24} seconds.

   a. Without using the theorem from today's lesson, explain why the neutral rho meson has a longer average lifetime.

   Since $3 \times 10^{-25} = 3 \times 10^{-1} \times 10^{-24}$, we can compare $3 \times 10^{-1} \times 10^{-24}$ and 4.5×10^{-24}. Based on Example 3 or by use of (1) above, we only need to compare 3×10^{-1} and 4.5, which is the same as 0.3 and 4.5. If we multiply each number by 10, we get whole numbers 3 and 45. Since $3 < 45$, then $3 \times 10^{-25} < 4.5 \times 10^{-24}$. Therefore, the neutral rho meson has a longer average lifetime.

   b. Approximately how much longer is the lifetime of a neutral rho meson than a Z boson?

   $45:3$ *or* 15 *times longer*

Rapid White Board Exchange: Operations with Numbers Expressed in Scientific Notation

1. $(5 \times 10^4)^2 =$

   $\mathbf{2.5 \times 10^9}$

2. $(2 \times 10^9)^4 =$

   $\mathbf{1.6 \times 10^{37}}$

3. $\frac{(1.2\times10^4)+(2\times10^4)+(2.8\times10^4)}{3} =$

   $\mathbf{2 \times 10^4}$

4. $\frac{7\times10^{15}}{14\times10^9} =$

   $\mathbf{5 \times 10^5}$

5. $\frac{4\times10^2}{2\times10^8} =$

   $\mathbf{2 \times 10^{-6}}$

6. $\frac{(7\times10^9)+(6\times10^9)}{2} =$

   $\mathbf{6.5 \times 10^9}$

7. $(9 \times 10^{-4})^2 =$

   $\mathbf{8.1 \times 10^{-7}}$

8. $(9.3 \times 10^{10}) - (9 \times 10^{10}) =$

   $\mathbf{3 \times 10^9}$