Lesson 14: Modeling with Inverse Trigonometric Functions

Student Outcomes

- Students model situations using trigonometric functions and apply inverse trigonometric functions to solve problems in modeling contexts.

Lesson Notes

In the previous lesson, students explored how inverse trigonometric functions could be used to solve best viewing problems. Throughout the lesson, students have opportunities to model real-world situations using trigonometric equations (MP.4) and solve the equations using inverse trigonometric functions. They apply inverse trigonometric functions to determine the angle of elevation for inclined surfaces and to solve problems involving periodic phenomena.

Classwork

Opening (5 minutes)

Ask students to imagine that they have been hired to construct a ramp from the ground to reach a door on a building that is 20 feet above the ground. To minimize construction costs, the slanted part of the ramp cannot be longer than 250 feet. Have students discuss in pairs how they could determine the minimum steepness of the ramp they have been commissioned to make. The pairs should produce a sketch to illustrate their thinking. After a few minutes, a few pairs could display their sketches and discuss their solutions, including the use of inverse trigonometric functions to determine the minimum steepness of the ramp. Students should be told that the methods they used to model the steepness of the ramp can be applied in a variety of modeling contexts, which are explored in this lesson.

Example 1 (10 minutes)

This example demonstrates how inverse trigonometric functions can be used to help architects design ramps to meet safety specifications. Part (a) should be completed in pairs, or if students are unfamiliar with force diagrams, it could be completed as part of a teacher-led discussion. If students complete the sketches in pairs, the sketches should be discussed in a whole-class setting or approved by the teacher to resolve any confusion before the rest of the problem is completed. Parts (b) and (c) should be completed in pairs.

Note: Students might need to be told that the unit newton is equivalent to \( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \).

- What features should be included in our sketch of the scale-model ramp?
  - Answers should address that a right triangle could be used to represent the ramp, with the hypotenuse representing the inclined ramp surface and the angle of elevation representing the acute angle between the inclined surface and the ground; the model should also include a circle to represent an object rolling down the ramp and vectors to represent the force applied to the ball and the acceleration of the ball down the ramp.
How can we represent the force of gravity applied to the ball on the ramp?
- It is a vector facing downward because the force of gravity is always directed perpendicular to the surface of the ground.

How can we represent the motion of the ball on the ramp?
- Answers may address drawing a vector parallel to the ramp pointed downward to represent the acceleration of the ball, which is 2.4 m/s².

What is causing the ball to accelerate down the ramp?
- The force due to gravity.

That’s right. However, our force vector is directed downward, not in the direction of the ball’s motion. How could we represent the force vector in a way that demonstrates its effect on the ball’s motion?
- Answers will vary but should address that we could decompose the force vector into a component that is parallel to the ramp and facing downward and a component that is perpendicular to the ramp.

If we sketch the vector’s components parallel and perpendicular to the ramp, these component vectors and the vector representing the gravitational force create a right triangle. How can we find the measures of the acute angles of this triangle?
- Answers will vary but should address that since the gravitational force vector is parallel to the height of the right triangle that represents the ramp, the angle formed between the height of the triangle and the hypotenuse is congruent to the angle formed by the gravitational force vector and the component parallel to the ramp. Because of the triangle angle sum theorem, the measure of the other acute angles (one in each triangle) are congruent with measure θ.

What does θ represent?
- The angle of elevation.

How can we use our sketch to find the value of θ?
- Answers should address applying the inverse sine function to the ratio of the force vector parallel to the ramp and the gravitational force vector.

How do we determine the magnitude of the force parallel to the ramp, which is causing the ball to roll down the ramp?
- We were told that the force is the product of the object’s mass and its acceleration, so we can multiply the ball’s mass, 0.1 kg, by its acceleration, 2.4 m/s².

Scaffolding:
- Introduce students to the concept of the free-body diagram (FBD) as a means of illustrating the forces acting upon an object. Explain that the object of interest is drawn as a rectangle, and the forces acting upon the object are represented as vectors whose initial point is the center of the object. Demonstrate each of the following situations for students, discuss the forces acting on the objects, and have them represent the forces acting on the object using a free-body diagram.
- Drop a small object from waist height:

\[ F_g = mg \]

- Lay the same object on a flat surface:

\[ F_N = mg \]

- Lay the object on an inclined surface so that it slides down:

\[ F_f: \text{ frictional force, which is parallel and opposite in direction to the object’s motion.} \]

\[ F_g: \text{ gravitational force, which points towards the center of Earth.} \]

\[ F_N: \text{ normal or contact force, which is perpendicular to the contact surface.} \]
Example 1

A designer wants to test the safety of a wheelchair ramp she has designed for a building before constructing it, so she creates a scale model. To meet the city’s safety requirements, an object that starts at a standstill from the top of the ramp and rolls down should not experience an acceleration exceeding $2.4 \text{ m/s}^2$.

a. A ball of mass $0.1 \text{ kg}$ is used to represent an object that rolls down the ramp. As it is placed at the top of the ramp, the ball experiences a downward force due to gravity, which causes it to accelerate down the ramp. Knowing that the force applied to an object is the product of its mass and acceleration, create a sketch to model the ball as it accelerates down the ramp.

b. If the ball rolls at the maximum allowable acceleration of $2.4 \text{ m/s}^2$, what is the angle of elevation for the ramp?

$F_{\text{ramp}} = m_{\text{ball}} \times a_{\text{ball}} = 0.1 \text{ kg} \times 2.4 \text{ m/s}^2 = 0.24 \text{ N parallel to the ramp and directed down the ramp}$

$F = m_{\text{ball}} \times a_{\text{gravity}} = 0.1 \text{ kg} \times 9.8 \text{ m/s}^2 = 0.98 \text{ N directed down toward the ground}$

$\sin (\theta) = \frac{F_{\text{ramp}}}{F} = \frac{0.24 \text{ N}}{0.98 \text{ N}}$

$\sin^{-1}(\sin (\theta)) = \sin^{-1}\left(\frac{0.24 \text{ N}}{0.98 \text{ N}}\right) \approx 14.2^\circ$

The angle of elevation for the ramp is approximately $14.2^\circ$.

Scaffolding:
- Students who are familiar with mechanics can create the sketch without being cued about the relationship between force, mass, and acceleration.
- Have advanced students calculate the maximum angle of elevation given a frictional force directed up the ramp.
c. If the designer wants to exceed the safety standards by ensuring the acceleration of the object does not exceed \(2.0 \, \text{m/s}^2\) by how much will the maximum angle of elevation decrease?

\[
F_{\text{ramp}} = m_{\text{ball}} \times a_{\text{ball}} = 0.1 \, \text{kg} \times 2.0 \, \text{m/s}^2 = 0.2 \, \text{N parallel to the ramp and directed down the ramp}
\]

\[
F = m_{\text{ball}} \times a_{\text{gravity}} = 0.1 \, \text{g} \times 9.8 \, \text{m/s}^2 = 0.98 \, \text{N directed down toward the ground}
\]

\[
\sin (\theta) = \frac{F_{\text{ramp}}}{F} = \frac{0.2 \, \text{N}}{0.98 \, \text{N}}
\]

\[
\sin^{-1}(\sin (\theta)) = \sin^{-1}\left(\frac{0.2 \, \text{N}}{0.98 \, \text{N}}\right) \approx 11.8^\circ
\]

\text{The maximum angle of elevation would be approximately } 11.8^\circ, \text{ which is } 2.4 \text{ degrees less than the maximum angle permitted to meet the safety specification.}

d. How does the mass of the ball used in the scale model affect the value of } \theta \text{? Explain your response.}

\text{It doesn't. The mass of the ball is a common factor in the numerator and denominator in the ratio used to calculate } \theta.\

\textbf{Exercise 1 (5 minutes)}

This exercise should be completed in pairs. At an appropriate time, selected students should share their responses. Additional students should be allowed to suggest revisions to the model or suggest alternative approaches to solving the problem.

\textbf{Exercise 1}

A vehicle with a mass of 1,000 kg rolls down a slanted road with an acceleration of \(0.07 \, \text{m/s}^2\). The frictional force between the wheels of the vehicle and the wet concrete road is 2,800 newtons.

a. Sketch the situation.

b. What is the angle of elevation of the road?

\[
F_{\text{road}} = m_{\text{vehicle}} \times a_{\text{vehicle}} + F_{fr} = 1000 \, \text{kg} \times 0.07 \, \text{m/s}^2 = 70 \, \text{N} + 2800 \, \text{N} = 2870 \, \text{N}
\]

\[
F_{\text{gravity}} = m_{\text{vehicle}} \times a_{\text{gravity}} = 1000 \, \text{kg} \times 9.8 \, \text{m/s}^2 = 9800 \, \text{N}
\]

\[
\sin (\theta) = \frac{F_{\text{road}}}{F_{\text{gravity}}} = \frac{2870 \, \text{N}}{9800 \, \text{N}}
\]

\[
\sin^{-1}(\sin (\theta)) = \sin^{-1}\left(\frac{2870 \, \text{N}}{9800 \, \text{N}}\right) \approx 17^\circ
\]

The angle of elevation for the road is approximately \(17^\circ\).
Lesson 14: Modeling with Inverse Trigonometric Functions

This example demonstrates how inverse trigonometric functions can be used to solve problems addressing periodic phenomena. Students apply inverse trigonometric functions to create a function equation that enables them to predict the calendar dates that correspond to specific angles of solar declination. This example should be completed as part of a teacher-led discussion. Alternatively, students could complete the problem in pairs and share their responses in a whole-class setting.

- How can we determine the domain of the function modeled in the problem?
  - Since \( N \) represents a calendar date, its values are counting numbers between 1, which represents January 1, and 365, which represents December 31 (excluding leap years).

- What do we know about the range?
  - We know that the cosine function has a range between \(-1\) and \(1\), inclusive. Therefore, the expression
    \[-23.44° \left( \cos \left( \frac{360}{365} \right) \left( N + 10 \right) \right)\]
    has a maximum value of \(23.44°\) when \(\cos \left( \frac{360}{365} \right) \left( N + 10 \right) = -1\)
    and a minimum value of \(-23.44°\) when \(\cos \left( \frac{360}{365} \right) \left( N + 10 \right) = 1\).

- What first step should we take to write \( N \) as a function of \( \delta \)?
  - Divide by \(-23.44°\) to isolate the cosine function.

- What’s next?
  - Apply the inverse cosine function to both sides of the equation.

- How does this help us to isolate \( N \)?
  - The inverse function undoes the operation applied by the cosine function.

- Once we have isolated \( N \), explain in context the meaning of this function.
  - Answers will vary but should address that the function provides a calendar date output given a declination angle input.

- What is the domain of the equation that represents \( N \) as a function of \( \delta \)? Explain.
  - \(-23.44° \leq \delta \leq 23.44°\). Any other values produce a ratio \(\frac{\delta}{-23.44°} > 1\), where the inverse cosine function is undefined.

- The range is a bit trickier. What is the domain of the inverse cosine function?
  - \(0° \leq \theta \leq 180°\)
Which means that the expression $-10 + \left(\frac{365}{360}\right) \cos^{-1}\left(\frac{\delta}{-23.44}\right)$ is between which two values?

- $-10$ if $\cos^{-1}\left(\frac{\delta}{-23.44}\right) = 0^\circ$ and $172.5$ if $\cos^{-1}\left(\frac{\delta}{-23.44}\right) = 180^\circ$

These values account for about half the calendar dates. Clearly, solar declination exists for calendar dates greater than 172.5. How can we find appropriate dates in the second half of the calendar year that correspond to a given declination angle?

- *Answers will vary but might address using the property that* $\cos(360^\circ - x) = \cos(x)$.

How can we apply this to the declination angle $10^\circ$?

- *Answers will vary but might address that* $x \approx 115.3^\circ$ and also $(360^\circ - 115.3^\circ) \approx 244.7^\circ$, so both of these values could be used to determine $N$.

If we evaluate the function for $\cos^{-1}\left(\frac{10^\circ}{-23.44}\right) \approx 115.3^\circ$, we get $N \approx 107$. Why can’t we just subtract this $N$ value from 365 to find the other value of $N$?

- *Answers will vary but should address that the original function was phase shifted 10 units to the left.*

How can we determine the dates when the sun is directly overhead?

- *It is when the declination is $0^\circ$.*

When would you predict these dates to be?

- *Answers will vary, but some students might predict that this occurs at the spring and autumn equinoxes.*

Our model provided dates of March 22 and September 21 as outputs for a declination input of $0^\circ$. The dates of the equinoxes are known to be March 20 and September 22. What might account for the differences?

- *Answers will vary but might address issues such as not accounting for leap days to possible simplifications in the original function for ease of use.*

**Example 2**

The declination of the sun is the path the sun takes overhead the earth throughout the year. When the sun passes directly overhead, the declination is defined as $0^\circ$, while a positive declination angle represents a northward deviation and a negative declination angle represents a southward deviation. Solar declination is periodic and can be roughly estimated using the equation

$$\delta = -23.44^\circ \left(\cos\left(\frac{360}{365}(N + 10)\right)\right)$$

where $N$ represents a calendar date (e.g., $N = 1$ is January 1, and $\delta$ is the declination angle of the sun measured in degrees).

a. Describe the domain and range of the function.

- $D: 1 \leq N \leq 365$ where $N$ is a counting number
- $R: -23.44^\circ \leq \delta \leq 23.44^\circ$
b. Write an equation that represents \( N \) as a function of \( \delta \).

\[
\delta = -23.44^\circ \left( \cos \left( \frac{360}{365} (N + 10) \right) \right)
\]

\[
\frac{\delta}{-23.44^\circ} = \left( \cos \left( \frac{360}{365} (N + 10) \right) \right)
\]

\[
\cos^{-1} \left( \frac{\delta}{-23.44^\circ} \right) = \frac{360}{365} (N + 10)
\]

\[
-10 + \frac{365}{360} \cos^{-1} \left( \frac{\delta}{-23.44^\circ} \right) = N
\]

c. Determine the calendar date(s) for the given angles of declination:

i. \( 10^\circ \)

\[
N = -10 + \left( \frac{365}{360} \cos^{-1} \left( \frac{10^\circ}{-23.44^\circ} \right) \right) \\
N \approx 107 \text{ and } 238
\]

\( N = 107 \) corresponds to a calendar date of April 17, and \( N = 238 \) corresponds to August 26.

ii. \( -5.2^\circ \)

\[
N = -10 + \left( \frac{365}{360} \cos^{-1} \left( \frac{-5.2^\circ}{-23.44^\circ} \right) \right) \\
N \approx 68 \text{ and } 277
\]

\( N = 68 \) corresponds to a calendar date of March 9, and \( N = 277 \) corresponds to October 4.

iii. \( 25^\circ \)

No date will correspond to this angle because it lies outside of the domain of the function.

d. When will the sun trace a direct path above the equator?

When the sun passes directly overhead, the declination is \( 0^\circ \). This means that

\[
N = -10 + \left( \frac{365}{360} \cos^{-1} \left( \frac{0^\circ}{-23.44^\circ} \right) \right) \approx 81 \text{ and } 264, \text{ which correspond to the calendar dates}
\]

March 22 and September 21.

Exercises 2–3 (10 minutes)

These exercises should be completed in pairs. At an appropriate time, selected students should share their responses. If students struggle with the curve fitting in Exercise 3, this exercise could be completed as part of a teacher-led discussion.

Graphing calculators, graphing software, or grid paper are needed for students to complete Exercise 3.

Exercises 2–3

2. The average monthly temperature in a coastal city in the United States is periodic and can be modeled with the equation \( y = -8 \cos \left( \frac{x - 1}{6} \pi \right) + 17.5 \), where \( y \) represents the average temperature in degrees Celsius and \( x \) represents the month, with \( x = 1 \) representing January.

a. Write an equation that represents \( x \) as a function of \( y \).

\[
x = 1 + \frac{6}{\pi} \cos^{-1} \left( \frac{y - 17.5}{-8} \right)
\]
b. A tourist wants to visit the city when the average temperature is closest to 25°C Celsius. What recommendations would you make regarding when the tourist should travel? Justify your response.

\[ x = 1 + \frac{6}{\pi} \cos^{-1} \left( \frac{25 - 17.5}{8} \right) \approx 6.32 \]

If the tourist wants to visit when the temperature is closest to 25°C Celsius, she should travel about the second week in June.

3. The estimated size for a population of rabbits and a population of coyotes in a desert habitat are shown in the table. The estimated population sizes were recorded as part of a long-term study related to the effect of commercial development on native animal species.

<table>
<thead>
<tr>
<th>Years since initial count (n)</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>21</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimated number of rabbits (r)</td>
<td>14,989</td>
<td>10,055</td>
<td>5,002</td>
<td>10,033</td>
<td>15,002</td>
<td>10,204</td>
<td>4,999</td>
<td>10,002</td>
<td>14,985</td>
</tr>
<tr>
<td>Estimated number of coyotes (c)</td>
<td>1,995</td>
<td>2,201</td>
<td>2,003</td>
<td>1,795</td>
<td>1,999</td>
<td>2,208</td>
<td>2,010</td>
<td>1,804</td>
<td>2,001</td>
</tr>
</tbody>
</table>

a. Describe the relationship between sizes of the rabbit and coyote populations throughout the study.

The rabbit population started at approximately 15,000 rabbits and then decreased while the coyote population increased (perhaps because of the abundance of prey for the coyotes). Over time, both the rabbit and coyote populations declined until the coyote population was about 1,800, when the rabbit population increased again. Both species’ population numbers appear to cycle, with the coyotes’ values shifted about 3 years from the rabbit’s values.

b. Plot the relationship between the number of years since the initial count and the number of rabbits. Fit a curve to the data.

Equation of curve: \[ r = 5000 \cos \left( \frac{\pi n}{6} \right) + 10000 \]
c. Repeat the procedure described in part (b) for the estimated number of coyotes over the course of the study.

$$\text{Equation of curve: } c = 200 \sin \left( \frac{\pi n}{6} \right) + 2000$$

\begin{align*}
\text{d. During the study, how many times was the rabbit population approximately 12,000?} \\
\text{When were these times?} \\
\begin{align*}
r &= 5000 \cos \left( \frac{\pi n}{6} \right) + 10000 \\
n &= 6 \cos^{-1} \left( \frac{r - 10000}{5000} \right) \\
n &= 6 \cos^{-1} \left( \frac{12000 - 10000}{5000} \right) \approx 2.2
\end{align*}
\end{align*}

Given that the function cycles every 12 years, $n = 0 + 2.2 = 2.2$; $n = (12 - 2.2) = 9.8$; $n = (12 + 2.2) = 14.2$; and $n = (24 - 2.2) = 21.8$. The rabbit population was approximately 12,000 four times at 2.2, 9.8, 14.2, and 21.8 years.

\begin{align*}
\text{e. During the study, when was the coyote population estimate below 2,100?} \\
c &= 200 \sin \left( \frac{\pi n}{6} \right) + 2000 \\
n &= 6 \sin^{-1} \left( \frac{c - 2000}{200} \right) \\
n &= 6 \sin^{-1} \left( \frac{2100 - 2000}{200} \right) = 1.5, 13, 17
\end{align*}

By analyzing the graph of the coyote population estimates, the population of coyotes was less than 2,100 prior to $n = 1$, between $n = 5$ and $n = 13$, and from $n = 17$ to $n = 24$. 

Scaffolding:
- Cue the students to use their graphs to verify their solutions.
- Remind the students of the periodicity and symmetry of the sine and cosine functions, which lead to multiple values for $n$ within their domain.
Closing (5 minutes)

Have students summarize, in writing, the different contexts they have encountered that could be modeled with trigonometric functions. The summaries should include a brief discussion of how inverse trigonometric functions were applied in solving problems in the modeling contexts. Students should share their summaries with a partner.

- **Right triangle trigonometry can be applied to model the forces applied to an object on an inclined surface.**
- **Inverse trigonometric functions can be applied to undo the effects of the trigonometric operation, which, in the inclined surface problem, was used to determine the angle of elevation.**
- **Trigonometric functions may be used to represent periodic phenomena, such as the declination of the sun.**
- **Inverse trigonometric functions can be applied to represent the relationship between variables so that the input is modeled as a function of the output. For instance, when we were presented an equation that modeled declination as a function of calendar date, inverse trigonometric functions allowed us to model calendar date as a function of the solar declination angle.**

Exit Ticket (5 minutes)
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Exit Ticket

The minimum radius of the turn $r$ needed for an aircraft traveling at true airspeed $v$ is given by the following formula

$$r = \frac{v^2}{g \tan(\theta)}$$

where $r$ is the radius in meters, $g$ is the acceleration due to gravity, and $\theta$ is the banking angle of the aircraft. Use $g = 9.78 \text{ m/s}^2$ instead of $9.81 \text{ m/s}^2$ to model the acceleration of the airplane accurately at 30,000 ft.

a. If an aircraft is traveling at $103 \text{ m/s}$, what banking angle is needed to successfully turn within 1 km?

b. Write the formula that gives the banking angle as a function of the radius of the turn available for a fixed airspeed $v$.

c. For a variety of reasons, including motion sickness from fluctuating $g$-forces and the danger of losing lift, many airplanes have a maximum banking angle of around $60^\circ$. Does this maximum on the model affect the domain or range of the formula you gave in part (b)?
Exit Ticket Sample Solutions

The minimum radius of the turn $r$ needed for an aircraft traveling at true airspeed $v$ is given by the following formula

$$r = \frac{v^2}{g \tan(\theta)}$$

where $r$ is the radius in meters, $g$ is the acceleration due to gravity, and $\theta$ is the banking angle of the aircraft. Use $g = 9.78 \frac{m}{s^2}$ instead of $9.81 \frac{m}{s^2}$ to model the acceleration of the airplane accurately at 30,000 ft.

a. If an aircraft is traveling at $0.3 \frac{m}{s}$, what banking angle is needed to successfully turn within 1 km?

$$1000 = \frac{103^2}{9.78 \cdot \tan(\theta)}$$

$$\theta = \tan^{-1}\left(\frac{103^2}{1000 \cdot 9.78}\right)$$

$$\approx 47.328$$

About 47.3°

b. Write the formula that gives the banking angle as a function of the radius of the turn available for a fixed airspeed $v$.

$$\theta = \tan^{-1}\left(\frac{v^2}{r \cdot g}\right)$$

c. For a variety of reasons, including motion sickness from fluctuating $g$-forces and the danger of losing lift, many airplanes have a maximum banking angle of around 60°. Does this maximum on the model affect the domain or range of the formula you gave in part (b)?

*If the angle cannot be greater than 60°, then the range caps out at 60° instead of normally being able to go up to 90°.*

Problem Set Sample Solutions

1. A particle is moving along a line at a velocity of $y = 3 \sin\left(\frac{2\pi x}{5}\right) + 2 \frac{m}{s}$ at location $x$ meters from the starting point on the line for $0 \leq x \leq 20$.

   a. Find a formula that represents the location of the particle given its velocity.

   $$y = \frac{5}{2\pi} \cdot \sin^{-1}\left(\frac{x - 2}{3}\right)$$

   b. What is the domain and range of the function you found in part (a)?

   *The domain is \(-1 \leq x \leq 5\), and the range is \(-\frac{5}{4} \leq y \leq \frac{5}{4}.*

   c. Use your answer to part (a) to find where the particle is when it is traveling $5 \frac{m}{s}$ for the first time.

   *The particle will be located at $x = 1.2$ meters from the starting point on the line.*
d. How can you find the other locations the particle is traveling at this speed?

In this case, the velocity is a maximum, so it will only occur once every period. All other values can be found by adding multiples of 5 to the location. If it was not a maximum, we could subtract the location from $\frac{5}{2}$ to find another value within the same period and then add multiples of 5 to find analogous values in other periods.

2. In general, since the cosine function is merely the sine function under a phase shift, mathematicians and scientists regularly choose to use the sine function to model periodic phenomena instead of a mixture of the two. What behavior in data would prompt a scientist to use a tangent function instead of a sine function?

The tangent function has infinitely many vertical asymptotes and rapidly takes on extreme values. Since the tangent function is the ratio between the sine and cosine functions, it will probably show up when comparing the ratio of two sets of periodic data. Otherwise, the extreme values would be reasons to use the tangent function.

3. A vehicle with a mass of 500 kg rolls down a slanted road with an acceleration of $0.04 \, \text{m} / \text{s}^2$. The frictional force between the wheels of the vehicle and the road is 1,800 newtons.

a. Sketch the situation.

b. What is the angle of elevation of the road?

\[
F_{\text{road}} = ma + F_{\text{fr}} = 500 \, \text{kg} \times 0.04 \, \text{m} / \text{s}^2 + 1800 \, \text{N} = 20 \, \text{N} + 1800 \, \text{N} = 1820 \, \text{N}
\]

\[
F_g = mg = 500 \, \text{kg} \times 9.8 \, \text{m} / \text{s}^2 = 4900 \, \text{N}
\]

\[
\sin(\theta) = \frac{F_{\text{road}}}{F_g} = \frac{1820 \, \text{N}}{4900 \, \text{N}} \approx 0.371
\]

\[
\sin^{-1}(\sin(\theta)) = \sin^{-1}(0.371) \approx 21.8^\circ
\]

The angle of elevation for the road is approximately 21.8°.
c. The steepness of a road is frequently measured as grade, which expresses the slope of a hill as a ratio of the change in height to the change in the horizontal distance. What is the grade of the hill described in this problem?

We need to find the perpendicular force. We get

\[ \cos(21.8) = \frac{F_{\text{perp}}}{4900} \]

\[ F_{\text{perp}} \approx 4549.46. \]

So the grade of the hill is \( \frac{1820}{4549.46} \approx 40\% \).

4. Canton Avenue in Pittsburgh, PA is considered to be one of the steepest roads in the world with a grade of 37\%.

a. Assuming no friction on a particularly icy day, what would be the acceleration of a 1,000 kg car with only gravity acting on it?

\[ \tan(\theta) = 0.37 \]
\[ \theta \approx 20.30^\circ \]

Since the acceleration due to gravity is the only acceleration on the car, the acceleration due to gravity is being transferred into an acceleration as the car goes down the hill. We can envision the force due to gravity as two separate forces, one that is parallel to the road and the second that is perpendicular to the road. The force parallel to the road is \( F_\parallel = m \cdot g \cdot \sin(\theta) \), and the acceleration parallel to the road is

\[ a = g \cdot \sin(\theta) \]
\[ a \approx 9.8 \sin(20.3) \]
\[ a \approx 3.4 \]

The acceleration is \( 3.4 \, \text{m/s}^2 \).

b. The force due to friction is equal to the product of the force perpendicular to the road and the coefficient of friction \( \mu \). For icy roads and a non-moving vehicle, assume the coefficient of friction is \( \mu = 0.3 \). Find the force due to friction for the car above. If the car is in park, will it begin sliding down Canton Avenue if the road is this icy?

The force perpendicular to the road is 9191.3 N. \( F_\perp = m \cdot g \cdot \cos(\theta) = 1000 \cdot 9.8 \cdot \cos(20.3) \approx 9191.3 \)

Thus, the force due to friction is approximately 2757.4 N. \( \mu F_\perp \approx 2757.4 \).

The force parallel to the road is 3400.0 N. \( F_\parallel = 1000 \cdot 9.8 \cdot \sin(20.3) \approx 3400.0 \).

Because the force parallel to the road is greater than the force due to friction, the car will slide down the road once Canton Avenue gets this icy.

c. Assume the coefficient of friction for moving cars on icy roads is \( \mu = 0.2 \). What is the maximum angle of road that the car will be able to stop on?

The car will be able to slow (and eventually stop) when the force due to friction is greater than the force parallel to the road, so we need to solve,

\[ 9800 \cdot \sin(\theta) = 0.2 \cdot 9800 \cos(\theta) \]
\[ \frac{\sin(\theta)}{\cos(\theta)} = 0.2 \]
\[ \tan(\theta) = 0.2. \]

So the car will be able to slow on any hill with less than a 20\% grade, which corresponds to an angle of 11.3°.
5. Talladega Superspeedway has some of the steepest turns in all of NASCAR. The main turns have a radius of about 3.05 m and are pitched at 33°. Let \( N \) be the perpendicular force on the car and \( N_v \) and \( N_h \) be the vertical and horizontal components of this force, respectively. See the diagram below.

### a. Let \( \mu \) represent the coefficient of friction; recall that \( \mu N \) gives the force due to friction. To maintain the position of a vehicle traveling around the bank, the centripetal force must equal the horizontal force in the direction of the center of the track. Add the horizontal component of friction to the horizontal component of the perpendicular force on the car to find the centripetal force. Set your expression equal to \( \frac{mv^2}{r} \), the centripetal force.

The force due to friction is \( \mu N \), and the horizontal component then is \( \mu N \cos(\theta) \).

The horizontal perpendicular force would be \( N \sin(\theta) \).

We get, \( \frac{mv^2}{r} = N \sin(\theta) + \mu N \cos(\theta) \).

### b. Add the vertical component of friction to the force due to gravity, and set this equal to the vertical component of the perpendicular force.

\( N \cos(\theta) \) is the vertical component of the perpendicular force.

The force due to gravity is \( mg \), and the vertical force of friction is \( \mu N \sin(\theta) \).

We get, \( N \cos(\theta) = mg + \mu N \sin(\theta) \).
c. Solve one of your equations in part (a) or part (b) for \( m \), and use this with the other equation to solve for \( v \).

\[
\frac{m}{g} = \frac{N \cos(\theta) - \mu N \sin(\theta)}{N \sin(\theta) + \mu N \cos(\theta)}
\]

\[
\frac{v^2}{r} = \frac{N \sin(\theta) + \mu N \cos(\theta)}{N \cos(\theta) - \mu N \sin(\theta)}
\]

\[
\frac{v^2}{gr} = \frac{N \sin(\theta) + \mu N \cos(\theta)}{N \cos(\theta) - \mu N \sin(\theta)}
\]

We can factor out the \( N \) and cancel the common factor from here onward.

\[
v^2 = gr \cdot \frac{\sin(\theta) + \mu \cos(\theta)}{\cos(\theta) - \mu \sin(\theta)}
\]

\[
v = \sqrt{gr \cdot \frac{\sin(\theta) + \mu \cos(\theta)}{\cos(\theta) - \mu \sin(\theta)}}
\]

d. Assume \( \mu = 0.75 \), the standard coefficient of friction for rubber on asphalt. For the Talladega Superspeedway, what is the maximum velocity on the main turns? Is this about how fast you might expect NASCAR stock cars to travel? Explain why you think NASCAR takes steps to limit the maximum speeds of the stock cars.

\[
v = \sqrt{9.8 \cdot 305 \cdot \frac{\sin(33^\circ) + 0.75 \cos(33^\circ)}{\cos(33^\circ) - 0.75 \sin(33^\circ)}}
\]

\[
= 90.3 \text{ m/s}
\]

\( 90.3 \text{ m/s} \) is about 202 mph, which is the speed most people associate with NASCAR stock cars (although they usually go slower than this). This means that at Talladega, the race cars are able to go at their maximum speeds through the main turns.

If the cars go any faster than this, then they would drift up toward the wall, which may prompt them to oversteer and possibly spin out of control. NASCAR may limit the speeds of the cars because the racetracks themselves are not designed for cars that can go faster.

e. Does the friction component allow the cars to travel faster on the curve or force them to drive slower? What is the maximum velocity if the friction coefficient is zero on the Talladega roadway?

The friction component is in the direction toward the center of the racetrack (away from the walls), so it allows them to travel faster more safely. If \( \mu = 0 \), the equation becomes \( v = \sqrt{gr \tan(\theta)} \). Therefore, the maximum velocity is about 44.1 \( \text{m/s} \) or 99 mph.

f. Do cars need to travel slower on a flat roadway making a turn than on a banked roadway? What is the maximum velocity of a car traveling on a 305 m turn with no bank?

They need to travel much slower on a flat roadway making a turn than when traveling on a banked roadway. The normal force does not prevent the car from spinning out of control the way it does on a banked turn.

If \( \theta = 0 \), then the equation becomes \( v = \sqrt{gr \mu} \). Therefore, the maximum velocity is about 47.3 \( \text{m/s} \) or 106 mph, which is much slower than the velocity that cars can travel on the banked turns (202 mph).
6. At a particular harbor over the course of 24 hours, the following data on peak water levels was collected (measurements are in feet above the MLLW):

<table>
<thead>
<tr>
<th>Time</th>
<th>1:30</th>
<th>7:30</th>
<th>14:30</th>
<th>20:30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water Level</td>
<td>-0.211</td>
<td>8.21</td>
<td>-0.619</td>
<td>7.518</td>
</tr>
</tbody>
</table>

a. What appears to be the average period of the water level?

*It takes 13 hours to get from the first low point to the second, and 13 hours to get from the first high point to the second, so the average period is \( \frac{13 + 13}{2} = 13 \).*

b. What appears to be the average amplitude of the water level?

*There are three areas we can examine to get amplitudes, from 1:30 to 7:30, 7:30 to 14:30, and 14:30 to 20:30. We get amplitudes of \( \frac{8.21 - (-0.211)}{2} = 4.205 \), \( \frac{8.21 - (-0.619)}{2} = 4.4145 \), and \( \frac{7.518 - (-0.619)}{2} = 4.0685 \). We get 4.231 as the average amplitude.*

c. What appears to be the average midline for the water level?

3.748

d. Fit a curve of the form \( y = A \sin(\omega(x - h)) + k \) or \( y = A \cos(\omega(x - h)) + k \) modeling the water level in feet as a function of the time.

*Since it would make our curve more inaccurate to guess at what point the water levels will cross the midline, we can either start at 1:30 or 7:30 and use the cosine function. For 7:30, we get \( y = 4.231 \cos\left(\frac{2\pi}{13}(x - 7.5)\right) + 3.748 \).*

e. According to your function, how many times per day will the water level reach its maximum?

*It should reach its maximum levels twice a day usually, but there is the rare possibility that it will reach its maximum only once.*

f. How can you find other time values for a particular water level after finding one value from your function?

*The values repeat every 13 hours, so immediately once a value is found, adding any multiple of 13 will give other values that work. Additionally, if you have the inverse cosine value for a particular water level (but have not solved for \( x \) yet), then take the opposite, solve normally, and you will have another time to which you can add multiples of 13.*

g. Find the inverse function associated with the function in part (d). What is the domain and range of this function? What type of values does this function output?

\[
y = \frac{13}{2\pi} \cos^{-1}\left(\frac{x - 3.748}{4.231}\right) + 7.5
\]

*The domain is all real numbers \( x \), such that \(-0.483 \leq x \leq 7.979\), and the range is all real numbers \( y \) such that \(7.5 \leq y \leq 14\).*