Lesson 2: Properties of Trigonometric Functions

Student Outcomes

- Students explore the symmetry and periodicity of trigonometric functions.
- Students derive relationships between trigonometric functions using their understanding of the unit circle.

Lesson Notes

In the previous lesson, students reviewed the characteristics of the unit circle and used them to evaluate trigonometric functions for rotations of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians. They then explored the relationships between the trigonometric functions for rotations $\theta$ in all four quadrants and derived formulas to evaluate sine, cosine, and tangent for rotations $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$. In this lesson, we revisit the idea of periodicity of the trigonometric functions as introduced in Algebra II Module 1 Lesson 1. Students continue to explore the relationship between trigonometric functions for rotations $\theta$, examining the periodicity and symmetry of the sine, cosine, and tangent functions. They also use the unit circle to explore relationships between the sine and cosine functions.

Classwork

Opening Exercise (3 minutes)

Students studied the graphs of trigonometric functions extensively in Algebra II but may not instantly recognize the graphs of these functions. The Opening Exercise encourages students to relate the graph of the sine, cosine, and tangent functions to rotations of a ray studied in the previous lesson. At the end of this exercise, allow students to provide justification for how they matched the graphs to the functions.
Opening Exercise

The graphs below depict four trigonometric functions. Identify which of the graphs are \( f(x) = \sin(x) \), \( g(x) = \cos(x) \), and \( h(x) = \tan(x) \). Explain how you know.

![Graphs of sine, cosine, and tangent functions]

The first graph is the graph of the tangent function \( h(x) = \tan(x) \) because the range of the tangent function is all real numbers, and \( \tan(0) = 0 \), which rules out the third graph as a possibility.

The second graph (upper right) is the graph of the cosine function \( g(x) = \cos(x) \) because the range of the cosine function is \([-1, 1]\), and \( \cos(0) = 1 \), ruling out the fourth graph as a possibility.

The fourth graph (bottom right) is the graph of the sine function \( f(x) = \sin(x) \) because the range of the sine function is \([-1, 1]\), and \( \sin(0) = 0 \).

The third graph (bottom left) is the graph of the cotangent function. (This could be an extension problem for advanced students.)

Discussion (7 minutes): Periodicity of Sine, Cosine, and Tangent Functions

This discussion helps students relate their graphs of the sine, cosine, and tangent functions to the unit circle. Students use the unit circle to determine the periodicity of the sine, cosine, and tangent functions, and they apply the periodicity to evaluate trigonometric functions for values of \( \theta \) that are not between 0 and \( 2\pi \).
Let’s look at the graph of the function \( f(x) = \sin(x) \).

How would you describe this graph to someone who has not seen it? Share your response with a partner.

- Answers will vary but will probably address that there are \( x \)-intercepts at all integer multiples of \( \pi \), that the graph is a wave whose height oscillates between \(-1\) and \(1\), and that the cycle repeats every \( 2\pi \) radians.

Let’s look now at the graph of the function \( g(x) = \cos(x) \).

Compare and contrast this graph with the graph of \( f(x) = \sin(x) \). Share your thoughts with your partner.

- Answers will vary but will probably address that the graph of \( g(x) \) appears to be the same as that of \( f(x) \) shifted to the left by \( \frac{\pi}{2} \) radians. In other words, the range of the function and the periodicity are the same for both graphs, but the \( x \)-intercepts are different.

The sketches illustrate that both the sine and cosine functions are periodic. Turn to your partner and describe what you remember about periodic functions in general and about the sine and cosine functions in particular.

- Periodic functions repeat the same pattern every period; for the sine and cosine functions, the period is \( 2\pi \).

How can we use the paper plate model of the unit circle to explain the periodicity of the sine and cosine functions? Discuss your reasoning in terms of the position of the rider on the carousel.

- Answers may vary but should address that for any given rotation \( \theta \), the position of the rider on the carousel is \( (x_\theta, y_\theta) \). If we want to represent the rotation \( 2\pi + \theta \), we would rotate the plate one full turn counterclockwise, and the position of the rider would again be \( (x_\theta, y_\theta) \). This trend would continue for each additional rotation of \( 2\pi \) radians since a rotation of \( 2\pi \) radians represents a complete turn.
How does periodicity apply to negative rotational values (e.g., $\theta - 2\pi$)? Again, explain your reasoning in terms of the position of the rider on the carousel.

- It would not affect the position of the rider. The rotation $\theta - 2\pi$ represents a full turn backwards (clockwise) from the position $(x_{\theta}, y_{\theta})$, and the rider’s position will still be the same as for a rotation $\theta$.

And how does the pattern in the rider’s position on the carousel relate to the periodicity of the sine and cosine functions?

- We can see that both $x_{\theta}$ and $y_{\theta}$ repeat for every $2\pi$ radians of rotation. Since $\sin(\theta) = y_{\theta}$ and $\cos(\theta) = x_{\theta}$, we can conclude that $\sin(\theta) = \sin(2\pi n + \theta)$ and $\cos(\theta) = \cos(2\pi n + \theta)$ for all integer values of $n$.

We’ve determined that both $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are periodic, and we’ve found formulas to describe the periodicity. What about $h(x) = \tan(x)$? Let’s look at the graph of this function.

Is the tangent function also periodic? Explain how you know.

- Yes. Since the tangent function is the quotient of the sine and cosine functions, and the sine and cosine functions are periodic with the same period, the tangent function is also periodic.

How does the period of the tangent function compare with the periods of the sine and cosine functions?

- The tangent function has a period of $\pi$ radians, but the sine and cosine functions have periods of $2\pi$ radians.

Let’s see if we can use the unit circle to explain this discrepancy. For rotation $\theta$, how do we define $\tan(\theta)$?

- $\tan(\theta) = \frac{y_{\theta}}{x_{\theta}}$

Recall that in the previous lesson, the point on the unit circle that corresponded to a given rotation represented the position of a rider on a carousel with radius 1 unit. What does the ratio $\frac{y_{\theta}}{x_{\theta}}$ represent in terms of the position of the rider?

- It is the ratio of the front/back position to the right/left position for rotation $\theta$. 
Let’s examine this ratio as the carousel rotates. Use your unit circle model to examine the position of the carousel rider for a complete rotation. Use the front/back and right/left positions of the rider to determine how the value of \( \tan \theta \) changes as \( \theta \) increases from 0 to \( 2\pi \) radians. Share your findings with a partner.

Answers will vary but should address the following:

- The position of the rider starts at \((1,0)\), immediately to the right of center, at \( \theta = 0 \). Since \( \tan(\theta) = \frac{y}{x} \), \( \tan(0) = 0 \). As the carousel rotates to \( \frac{\pi}{2} \), the rider’s front/back position increases from 0 to 1, and the right/left position decreases from 1 to 0. This means that \( \tan \theta \) increases from 0 at \( \theta = 0 \) to infinity as \( \theta \) approaches \( \frac{\pi}{2} \).

- The position of the rider is at \((0,1)\) at \( \theta = \frac{\pi}{2} \). So, when \( \theta = \frac{\pi}{2} \), \( \tan(\theta) = \frac{1}{0} \) which is undefined. As the carousel rotates counterclockwise from \( \theta = \frac{\pi}{2} \), the rider’s front/back position decreases from 1 to 0, and the right/left position decreases from 0 to -1. As the carousel rotates from \( \theta = \frac{\pi}{2} \) to \( \pi \), \( \tan(\theta) \) increases from \(-\infty\) to 0.

- From \( \theta = \pi \) to \( \theta = \frac{3\pi}{2} \), the rider’s front/back position decreases from 0 to -1, and the right/left position increases from -1 to 0. This means that \( \tan \theta \) increases from 0 at \( \theta = \pi \) toward \(+\infty\) at \( \theta = \frac{3\pi}{2} \).

- From \( \theta = \frac{3\pi}{2} \) to \( \theta = 2\pi \), the rider’s front/back position increases from -1 to 0, and the right/left position increases from 0 to 1. This means that \( \tan(\theta) \) increases from \(-\infty\) at \( \theta = \frac{3\pi}{2} \) to 0 at \( \theta = 2\pi \).

- How do your findings relate to the periodicity of the tangent function?

  - The values for \( \tan(\theta) \) seem to repeat every \( \pi \) radians, as the sketch indicates.

- How can we explain this pattern based on our understanding of the unit circle and the definition of \( \tan(\theta) \)?

  - Answers will vary. An example of an acceptable response is shown: As our initial ray rotates through Quadrants I and III, the ratio of \( y_\theta \) to \( x_\theta \) is positive because the coordinates have the same sign. The ratio increases from 0 to \(+\infty\) because the magnitudes of the y-coordinates increase, while the magnitudes of the x-coordinates decrease toward 0. Similarly, in Quadrants II and IV, the ratio of \( y_\theta \) to \( x_\theta \) is negative because the coordinates have opposite signs. The ratio should increase from \(-\infty\) to 0 because the magnitudes of the y-coordinates decrease toward 0, while the magnitudes of the x-coordinates increase toward 1, which means the ratio of \( y_\theta \) to \( x_\theta \) becomes a smaller and smaller negative number until it reaches 0 at \( \theta = 0 \) and \( \theta = 2\pi \).

- Great observations. How can we formalize these thoughts about the periodicity of the tangent function using a formula?

  - \( \tan(\theta) = \tan(\theta + \pi n) \), where \( n \) is an integer.
Exercises 1–2 (5 minutes)

The following exercises reinforce the periodicity of the three primary trigonometric functions: sine, cosine, and tangent. Students should complete the exercises independently. After a few minutes, they could share their responses with a partner. Students could write their responses to Exercise 1 on individual white boards for quick checks or on paper. Exercise 2 should be discussed briefly in a whole-class setting.

Exercises 1–4

1. Use the unit circle to evaluate these expressions:
   a. \( \sin \left( \frac{17\pi}{4} \right) \)
      \[
      \sin \left( \frac{17\pi}{4} \right) = \sin \left( 4\pi + \frac{\pi}{4} \right) = \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}
      \]
   b. \( \cos \left( \frac{19\pi}{6} \right) \)
      \[
      \cos \left( \frac{19\pi}{6} \right) = \cos \left( 2\pi + \frac{7\pi}{6} \right) = \cos \left( \frac{7\pi}{6} \right) = \cos \left( \pi + \frac{\pi}{6} \right) = -\cos \left( \frac{\pi}{6} \right) = -\frac{\sqrt{3}}{2}
      \]
   c. \( \tan(450\pi) \)
      \[
      \tan(450\pi) = \tan(450\pi + 0) = \tan(0) = 0
      \]

2. Use the identity \( \sin(\pi + \theta) = -\sin(\theta) \) for all real-numbered values of \( \theta \) to verify the identity \( \sin(2\pi + \theta) = \sin(\theta) \) for all real-numbered values of \( \theta \).
   \[
   \sin(2\pi + \theta) = \sin(\pi + (\pi + \theta)) = -\sin(\pi + \theta) = -(-\sin(\theta)) = \sin(\theta)
   \]

Discussion (3 minutes): Symmetry of Sine, Cosine, and Tangent Functions

This discussion addresses how the properties of the unit circle reveal symmetry in the graphs of the trigonometric functions. Students determine the evenness/oddness of the graphs for sine and cosine, which they in turn use to determine the evenness/oddness of the tangent function.

Scaffolding:

- The identity \( \cos(\pi + \theta) = -\cos(\theta) \) can help students evaluate the function in Exercise 1 part (b).
- Prompt students by asking, “If we rewrite \( \cos \left( \frac{7\pi}{6} \right) \) as \( \cos(\pi + \theta) \), what is the value of \( \theta \) ?”

MP.7
Let’s look again at the graphs of the functions \( f(x) = \sin(x) \) and \( g(x) = \cos(x) \). Describe the symmetry of the graphs.

- The graph of \( f(x) = \sin(x) \) seems to be symmetric about the origin (or seems to have 180° rotational symmetry), and the graph of \( g(x) = \cos(x) \) seems to be symmetric about the y-axis.

Let’s explore these apparent symmetries using the unit circle. We recall that for a rotation \( \theta \) on our carousel, the rider’s position is defined as \((x_\theta, y_\theta)\). Describe the position of a rider if the carousel rotates \(-\theta \) radians.

- A rotation by \(-\theta \) is equivalent to a clockwise rotation by \( \theta \) radians, which results in a position of \((x_\theta, -y_\theta)\).
And how does this relate to our understanding of the symmetry of the sine and cosine functions?

- Since \( \sin(-\theta) = -y \), we have \( \sin(-\theta) = -\sin(\theta) \). This means that \( f(x) = \sin(x) \) is an odd function with symmetry about the origin. (If students do not suggest that the sine function is an odd function, remind them that an odd function is a function \( f \) for which \( f(-x) = -f(x) \) for any \( x \) in the domain of \( f \).

- Since \( \cos(-\theta) = x \), we have \( \cos(-\theta) = \cos(\theta) \). This means that \( g(x) = \cos(x) \) is an even function with symmetry about the \( y \)-axis. (If students do not suggest that the cosine function is an even function, remind them that an even function is a function \( f \) for which \( f(-x) = f(x) \) for any \( x \) in the domain of \( f \).

Let’s summarize these findings:

- For all real numbers \( \theta \), \( \sin(-\theta) = -\sin(\theta) \), and \( \cos(-\theta) = \cos(\theta) \).

Exercises 3–4 (5 minutes)

Students should complete the exercises independently. Students could write their responses to Exercise 3 on individual white boards for quick checks or on paper. A few selected students should share their responses for Exercise 4 in a whole-class setting.

Scaffolding:

- Remind struggling students of the identity \( \cos(2\pi - \theta) = \cos(\theta) \) to help them evaluate the function in Exercise 3 part (b).
- Have advanced students determine the symmetry for the graphs of the co-functions.

3. Use your understanding of the symmetry of the sine and cosine functions to evaluate these functions for the given values of \( \theta \).
   a. \( \sin\left(-\frac{\pi}{2}\right) \)

   \[ \sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 \]
b. \( \cos \left( -\frac{5\pi}{3} \right) \)

\[
\cos \left( -\frac{5\pi}{3} \right) = \cos \left( \frac{5\pi}{3} \right) = \cos \left( 2\pi - \frac{\pi}{3} \right) = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}
\]

4. Use your understanding of the symmetry of the sine and cosine functions to determine the value of \( \tan(-\theta) \) for all real-numbered values of \( \theta \). Determine whether the tangent function is even, odd, or neither.

\[
\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{-\sin(\theta)}{\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta)
\]

The tangent function is odd.

Exploratory Challenge/Exercises 5–6 (10 minutes)

This challenge requires students to use the properties of the unit circle to derive relationships between the sine and cosine functions. Students should complete the challenge in pairs or small groups, with each group completing either Exercise 5 or Exercise 6. After a few minutes, students should review their findings with other pairs or groups assigned to the same exercise. A few groups should then share their responses in a whole-class setting.

Exploratory Challenge/Exercises 5–6

5. Use your unit circle model to complete the table. Then use the completed table to answer the questions that follow.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \frac{\pi}{2} + \theta )</th>
<th>( \sin \left( \frac{\pi}{2} + \theta \right) )</th>
<th>( \cos \left( \frac{\pi}{2} + \theta \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \pi )</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( \frac{3\pi}{2} )</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>2( \pi )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>( \frac{5\pi}{2} )</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

a. What does the value \( \frac{\pi}{2} + \theta \) represent with respect to the rotation of the carousel?

It is a rotation by \( \theta \) radians counterclockwise from the starting point \( \frac{\pi}{2} \).

b. What pattern do you recognize in the values of \( \sin \left( \frac{\pi}{2} + \theta \right) \) as \( \theta \) increases from 0 to 2\( \pi \)?

Values of \( \sin \left( \frac{\pi}{2} + \theta \right) \) follow the same pattern as values of \( \cos(\theta) \).

c. What pattern do you recognize in the values of \( \cos \left( \frac{\pi}{2} + \theta \right) \) as \( \theta \) increases from 0 to 2\( \pi \)?

Values of \( \cos \left( \frac{\pi}{2} + \theta \right) \) follow the same pattern as values of \( -\sin(\theta) \).
Lesson 2: Properties of Trigonometric Functions

6. Use your unit circle model to complete the table. Then use the completed table to answer the questions that follow.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \frac{\pi}{2} - \theta )</th>
<th>( \sin \left( \frac{\pi}{2} - \theta \right) )</th>
<th>( \cos \left( \frac{\pi}{2} - \theta \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( -\frac{\pi}{2} )</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>( -\pi )</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>( -\frac{3\pi}{2} )</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

a. What does the value \( \frac{\pi}{2} - \theta \) represent with respect to the rotation of a rider on the carousel?

*It is a rotation by \( \theta \) radians clockwise from a point directly in front of the center of the carousel.*

b. What pattern do you recognize in the values of \( \sin \left( \frac{\pi}{2} - \theta \right) \) as \( \theta \) increases from 0 to \( 2\pi \)?

*Values of \( \sin \left( \frac{\pi}{2} - \theta \right) \) follow the same pattern as values of \( \cos(\theta) \).*

c. What pattern do you recognize in the values of \( \cos \left( \frac{\pi}{2} - \theta \right) \) as \( \theta \) increases from 0 to \( 2\pi \)?

*Values of \( \cos \left( \frac{\pi}{2} - \theta \right) \) follow the same pattern as values of \( \sin(\theta) \).*

d. Fill in the blanks to formalize these relationships:

\[
\sin \left( \frac{\pi}{2} + \theta \right) =
\]

\[
\cos \left( \frac{\pi}{2} + \theta \right) =
\]

\[
\sin \left( \frac{\pi}{2} + \theta \right) = \cos(\theta)
\]

\[
\cos \left( \frac{\pi}{2} + \theta \right) = -\sin(\theta)
\]
Lesson 2: Properties of Trigonometric Functions

Exercise 7 (5 minutes)

Have students complete this exercise independently and then verify the solutions with a partner. Students should share their approaches to solving the problems as time permits.

Exercise 7

7. Use your understanding of the relationship between the sine and cosine functions to verify these statements.
   a. \( \cos \left( \frac{4\pi}{3} \right) = \sin \left( -\frac{\pi}{6} \right) \)
      \[
      \cos \left( \frac{4\pi}{3} \right) = \cos \left( \frac{\pi}{2} + \frac{5\pi}{6} \right) = -\sin \left( \frac{5\pi}{6} \right) = -\sin \left( \pi - \frac{5\pi}{6} \right) = -\sin \left( \frac{\pi}{6} \right) = \sin \left( -\frac{\pi}{6} \right)
      \]
   b. \( \cos \left( \frac{5\pi}{4} \right) = \sin \left( \frac{7\pi}{4} \right) \)
      \[
      \cos \left( \frac{5\pi}{4} \right) = \cos \left( \frac{\pi}{2} + \frac{3\pi}{4} \right) = -\sin \left( \frac{3\pi}{4} \right) = -\left( -\sin \left( \pi + \frac{3\pi}{4} \right) \right) = \sin \left( \frac{7\pi}{4} \right)
      \]

Closing (2 minutes)

Have students respond in writing to this prompt.

- Why do we only need to know values of \( \sin(\theta) \) and \( \cos(\theta) \) for \( 0 \leq \theta < 2\pi \) in order to find the sine or cosine of any real number?
  
  Because the sine and cosine functions are periodic with period \( 2\pi \), we know that \( \cos(\theta + 2\pi) = \cos(\theta) \) and \( \sin(\theta + 2\pi) = \sin(\theta) \) for any real number \( \theta \). Thus, if \( x \) is any real number, and \( x \geq 2\pi \), then we just subtract \( 2\pi \) as many times as is needed so that the result is between \( 0 \) and \( 2\pi \), and then we can evaluate the sine and cosine. Likewise, if \( x < 0 \), then we add \( 2\pi \) as many times as needed so that the result is between \( 0 \) and \( 2\pi \) and then evaluate sine and cosine.

Lesson Summary

For all real numbers \( \theta \) for which the expressions are defined,

\( \sin(\theta) = \sin(2\pi n + \theta) \) and \( \cos(\theta) = \cos(2\pi n + \theta) \) for all integer values of \( n \)

\( \tan(\theta) = \tan(\pi n + \theta) \) for all integer values of \( n \)

\( \sin(-\theta) = -\sin(\theta), \cos(-\theta) = \cos(\theta), \) and \( \tan(-\theta) = -\tan(\theta) \)

\( \sin \left( \frac{\pi}{2} + \theta \right) = \cos(\theta) \) and \( \cos \left( \frac{\pi}{2} + \theta \right) = -\sin(\theta) \)

\( \sin \left( \frac{\pi}{2} - \theta \right) = \cos(\theta) \) and \( \cos \left( \frac{\pi}{2} - \theta \right) = \sin(\theta) \)

Exit Ticket (5 minutes)
Lesson 2: Properties of Trigonometric Functions

Exit Ticket

1. From the unit circle given, explain why the cosine function is an even function with symmetry about the $y$-axis, and the sine function is an odd function with symmetry about the origin.

![Cosine and Sine Functions Image]

2. Use the unit circle to explain why $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ for $\theta$ as shown in the figure to the right.

![Unit Circle and Angle Image]
Exit Ticket Sample Solutions

1. From the unit circle given, explain why the cosine function is an even function with symmetry about the y-axis, and the sine function is an odd function with symmetry about the origin.

![Diagram of unit circle with point (x, y)](image)

Suppose we rotate the point \((1, 0)\) by \(-\theta\) radians, where \(\theta \geq 0\). This gives the same x-coordinate as rotating by \(\theta\) radians, so if \(\theta \geq 0\), we have \(\cos(-\theta) = \cos(\theta)\). Likewise, if \(\theta < 0\) and we rotate the point \((1, 0)\) by \(-\theta\) radians, then the resulting point has the same x-coordinate as rotation of \((1, 0)\) by \(\theta\) radians. Thus, if \(\theta < 0\) then \(\cos(-\theta) = \cos(\theta)\). Since \(\cos(-\theta) = \cos(\theta)\) for all real numbers \(\theta\), the cosine function is even and is symmetric about the y-axis.

Suppose we rotate the point \((1, 0)\) by \(-\theta\) radians, where \(\theta \geq 0\). This gives the opposite y-coordinate as rotating by \(\theta\) radians, so if \(\theta \geq 0\) we have \(\sin(-\theta) = -\sin(\theta)\). Likewise, if \(\theta < 0\) and we rotate the point \((1, 0)\) by \(-\theta\) radians, then the resulting point has the opposite y-coordinate as rotation of \((1, 0)\) by \(\theta\) radians. Thus, if \(\theta < 0\) then \(\sin(-\theta) = -\sin(\theta)\). Since \(\sin(-\theta) = -\sin(\theta)\) for all real numbers \(\theta\), the sine function is odd and is symmetric about the origin.

2. Use the unit circle to explain why \(\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)\) for \(\theta\) as shown in the figure to the right.

The point where the terminal ray intersects the unit circle after rotation by \(\theta\) has coordinates \((\cos(\theta), \sin(\theta))\). If we draw perpendicular lines from this point to the x-axis and y-axis, we create two triangles. The vertical legs of these triangles both have the same length. The triangle on the left shows that this length is \(\cos\left(\frac{\pi}{2} - \theta\right)\), while the triangle on the right shows that this length is \(\sin(\theta)\).

Thus, \(\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)\).

Problem Set Sample Solutions

1. Evaluate the following trigonometric expressions. Show how you used the unit circle to determine the solution.

   a. \(\sin\left(\frac{13\pi}{6}\right)\)

   \[
   \sin\left(\frac{13\pi}{6}\right) = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}
   \]

   \(\frac{13\pi}{6}\) is a full rotation more than \(\frac{\pi}{6}\) meaning \(\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\).
b. \( \cos \left( \frac{5\pi}{3} \right) \)

\[
\cos \left( \frac{5\pi}{3} \right) = \cos \left( -\left( 2\pi - \frac{\pi}{3} \right) \right) = \cos \left( -\frac{\pi}{3} \right) = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}
\]

\(-\frac{5\pi}{3}\) is a full rotation less than \(\frac{\pi}{3}\) meaning \(\cos \left( -\frac{5\pi}{3} \right) = \cos \left( \frac{\pi}{3} \right) = \frac{1}{2}\).

c. \( \tan \left( \frac{25\pi}{4} \right) \)

\[
\tan \left( \frac{25\pi}{4} \right) = \tan \left( 6\pi + \frac{\pi}{4} \right) = \tan \left( \frac{\pi}{4} \right) = 1
\]

\(\frac{25\pi}{4}\) is three full rotations more than \(\frac{\pi}{4}\) meaning \(\tan \left( \frac{25\pi}{4} \right) = \tan \left( \frac{\pi}{4} \right) = 1\).

d. \( \sin \left( -\frac{3\pi}{4} \right) \)

\[
\sin \left( -\frac{3\pi}{4} \right) = -\sin \left( \frac{3\pi}{4} \right) = -\sin \left( \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2}
\]

\(-\frac{3\pi}{4}\) is the same rotation as \(\pi + \frac{\pi}{4}\) meaning \(\sin \left( -\frac{3\pi}{4} \right) = -\sin \left( \frac{\pi}{4} \right) = -\frac{\sqrt{2}}{2}\).

e. \( \cos \left( -\frac{5\pi}{6} \right) \)

\[
\cos \left( -\frac{5\pi}{6} \right) = \cos \left( \frac{5\pi}{6} \right) = \cos \left( \pi - \frac{\pi}{6} \right) = -\cos \left( \frac{\pi}{6} \right) = -\frac{\sqrt{3}}{2}
\]

\(-\frac{5\pi}{6}\) is the same rotation as \(\pi + \frac{\pi}{6}\) meaning \(\cos \left( -\frac{5\pi}{6} \right) = -\cos \left( \frac{\pi}{6} \right) = -\frac{\sqrt{3}}{2}\).

f. \( \sin \left( \frac{17\pi}{3} \right) \)

\[
\sin \left( \frac{17\pi}{3} \right) = \sin \left( 4\pi + \frac{5\pi}{3} \right) = \sin \left( \frac{5\pi}{3} \right) = \sin \left( 2\pi - \frac{\pi}{3} \right) = -\sin \left( \frac{\pi}{3} \right) = -\frac{\sqrt{3}}{2}
\]

\(\frac{17\pi}{3}\) is two full rotations more than \(\frac{5\pi}{3}\) meaning \(\sin \left( \frac{17\pi}{3} \right) = \sin \left( \frac{5\pi}{3} \right) = -\frac{\sqrt{3}}{2}\).

g. \( \cos \left( \frac{25\pi}{4} \right) \)

\[
\cos \left( \frac{25\pi}{4} \right) = \cos \left( 6\pi + \frac{\pi}{4} \right) = \cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}
\]

\(\frac{25\pi}{4}\) is three full rotations more than \(\frac{\pi}{4}\) meaning \(\cos \left( \frac{25\pi}{4} \right) = \cos \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2}\).

h. \( \tan \left( \frac{29\pi}{6} \right) \)

\[
\tan \left( \frac{29\pi}{6} \right) = \tan \left( 4\pi + \frac{5\pi}{6} \right) = \tan \left( \frac{5\pi}{6} \right) = \tan \left( \pi - \frac{\pi}{6} \right) = -\tan \left( \frac{\pi}{6} \right) = -\frac{1}{\sqrt{3}}
\]

\(\frac{29\pi}{6}\) is two full rotations more than \(\frac{5\pi}{6}\) meaning \(\tan \left( \frac{29\pi}{6} \right) = -\tan \left( \frac{\pi}{6} \right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}\).
i. \( \sin \left( -\frac{31\pi}{6} \right) \)

\[
\sin \left( -\frac{31\pi}{6} \right) = -\sin \left( \frac{31\pi}{6} \right) = -\sin \left( 4\pi + \frac{7\pi}{6} \right) = -\sin \left( \frac{7\pi}{6} \right) = -\sin \left( \pi + \frac{\pi}{6} \right) = -\left( -\sin \left( \frac{\pi}{6} \right) \right) = \frac{1}{2}
\]

\(-\frac{31\pi}{6}\) is the same as two full rotations in the clockwise direction plus \(\frac{7\pi}{6}\) more, which is the same rotation as \(\frac{5\pi}{6}\) meaning \(\sin \left( -\frac{31\pi}{6} \right) = \sin \left( \frac{\pi}{6} \right) = \frac{1}{2}\).

j. \( \cos \left( -\frac{32\pi}{6} \right) \)

\[
\cos \left( -\frac{32\pi}{6} \right) = \cos \left( \frac{32\pi}{6} \right) = \cos \left( 4\pi + \frac{8\pi}{6} \right) = \cos \left( \frac{4\pi}{3} \right) = \cos \left( \pi + \frac{\pi}{3} \right) = -\cos \left( \frac{\pi}{3} \right) = -\frac{1}{2}
\]

\(-\frac{32\pi}{6}\) is the same as two full rotations in the clockwise direction plus \(\frac{8\pi}{6}\) or \(\frac{4\pi}{3}\) more, which is the same rotation as \(\pi + \frac{\pi}{3}\) meaning \(\cos \left( -\frac{32\pi}{6} \right) = -\cos \left( \frac{\pi}{3} \right) = -\frac{1}{2}\).

k. \( \tan \left( -\frac{18\pi}{3} \right) \)

\[
\tan \left( -\frac{18\pi}{3} \right) = \tan \left( 6\pi \right) = \tan (0) = 0
\]

\(-\frac{18\pi}{3}\) is three full rotations in the clockwise direction meaning \(\tan \left( -\frac{18\pi}{3} \right) = \tan (0) = 0\).

2. Given each value of \(\beta\) below, find a value of \(\alpha\) with \(0 \leq \alpha \leq 2\pi\) so that \(\cos(\alpha) = \cos(\beta)\) and \(\alpha \neq \beta\).

a. \(\beta = \frac{5\pi}{4}\)

\(\frac{5\pi}{4}\)

b. \(\beta = \frac{5\pi}{6}\)

\(\frac{7\pi}{6}\)

c. \(\beta = \frac{11\pi}{12}\)

\(\frac{13\pi}{12}\)

d. \(\beta = 2\pi\)

\(0\)

e. \(\beta = \frac{7\pi}{5}\)

\(\frac{3\pi}{5}\)
3. Given each value of $\beta$ below, find two values of $\alpha$ with $0 \leq \alpha \leq 2\pi$ so that $\cos(\alpha) = \sin(\beta)$.
   
   a. $\beta = \frac{\pi}{3}$
   \[
   \pi \quad \frac{11\pi}{6} \quad \frac{6}{6} \quad \frac{6}{6}
   \]
   
   b. $\beta = \frac{7\pi}{6}$
   \[
   2\pi \quad \frac{4\pi}{3} \quad \frac{3}{3} \quad \frac{3}{3}
   \]
   
   c. $\beta = \frac{3\pi}{4}$
   \[
   \pi \quad \frac{7\pi}{4} \quad \frac{4}{4} \quad \frac{4}{4}
   \]
   
   d. $\beta = \frac{\pi}{12}$
   \[
   3\pi \quad \frac{13\pi}{8} \quad \frac{8}{8} \quad \frac{8}{8}
   \]

4. Given each value of $\beta$ below, find two values of $\alpha$ with $0 \leq \alpha \leq 2\pi$ so that $\sin(\alpha) = \cos(\beta)$.
   
   a. $\beta = \frac{\pi}{3}$
   \[
   \pi \quad \frac{5\pi}{6} \quad \frac{6}{6} \quad \frac{6}{6}
   \]
   
   b. $\beta = \frac{5\pi}{6}$
   \[
   4\pi \quad \frac{5\pi}{3} \quad \frac{3}{3} \quad \frac{3}{3}
   \]
   
   c. $\beta = \frac{7\pi}{4}$
   \[
   \pi \quad \frac{3\pi}{4} \quad \frac{4}{4} \quad \frac{4}{4}
   \]
   
   d. $\beta = \frac{\pi}{12}$
   \[
   5\pi \quad \frac{7\pi}{12} \quad \frac{12}{12} \quad \frac{12}{12}
   \]
5. Jamal thinks that \( \cos \left( \alpha - \frac{\pi}{4} \right) = \sin \left( \alpha + \frac{\pi}{4} \right) \) for any value of \( \alpha \). Is he correct? Explain how you know.

   Jamal is correct. Let \( \theta = \alpha + \frac{\pi}{4} \). Then \( \theta - \frac{\pi}{2} = \alpha - \frac{\pi}{4} \). We know that \( \cos \left( \frac{\pi}{2} - \theta \right) = \sin(\theta) \) and that the cosine function is even, so we have \( \cos \left( \theta - \frac{\pi}{2} \right) = \sin(\theta) \). Then \( \cos \left( \alpha - \frac{\pi}{4} \right) = \sin \left( \alpha + \frac{\pi}{4} \right) \).

6. Shawna thinks that \( \cos \left( \alpha - \frac{\pi}{3} \right) = \sin \left( \alpha + \frac{\pi}{6} \right) \) for any value of \( \alpha \). Is she correct? Explain how you know.

   Shawna is correct. Let \( \theta = \alpha + \frac{\pi}{6} \). Then \( \theta - \frac{\pi}{2} = \alpha - \frac{\pi}{3} \). We know that \( \cos \left( \frac{\pi}{2} - \theta \right) = \sin(\theta) \) and that the cosine function is even, so we have \( \cos \left( \theta - \frac{\pi}{2} \right) = \sin(\theta) \). Then \( \cos \left( \alpha - \frac{\pi}{3} \right) = \sin \left( \alpha + \frac{\pi}{6} \right) \).

7. Rochelle looked at Jamal and Shawna’s results from Problems 5 and 6 and came up with the conjecture below. Is she correct? Explain how you know.

   Conjecture: \( \cos(\alpha - \beta) = \sin \left( \alpha + \left( \frac{\pi}{2} - \beta \right) \right) \).

   Rochelle is also correct. Because \( \sin \left( \theta + \frac{\pi}{2} \right) = \cos(\theta) \), we see that \( \sin \left( \alpha + \left( \frac{\pi}{2} - \beta \right) \right) = \sin \left( \alpha - \beta + \frac{\pi}{2} \right) = \cos(\alpha - \beta) \).

8. A frog is sitting on the edge of a playground carousel with radius 1 meter. The ray through the frog’s position and the center of the carousel makes an angle of measure \( \theta \) with the horizontal, and his starting coordinates are approximately \((0.81, 0.59)\). Find his new coordinates after the carousel rotates by each of the following amounts.

   a. \( \frac{\pi}{2} \)
      
      \[
      \cos \left( \theta + \frac{\pi}{2} \right) = -\sin(\theta) = -0.59 \\
      \sin \left( \theta + \frac{\pi}{2} \right) = \cos(\theta) = 0.81 \\
      \text{New position: } (-0.59, 0.81)
      \]

   b. \( \pi \)
      
      \[
      \cos(\theta + \pi) = -\cos(\theta) = -0.81 \\
      \sin(\theta + \pi) = -\sin(\theta) = -0.59 \\
      \text{New position: } (-0.81, -0.59)
      \]

   c. \( 2\pi \)
      
      \[
      \cos(\theta + 2\pi) = \cos(\theta) = 0.81 \\
      \sin(\theta + 2\pi) = \sin(\theta) = 0.59 \\
      \text{New position: } (0.81, 0.59)
      \]

   d. \( -\frac{\pi}{2} \)
      
      \[
      \cos \left( \theta - \frac{\pi}{2} \right) = \sin(\theta) = 0.59 \\
      \sin \left( \theta - \frac{\pi}{2} \right) = -\cos(\theta) = -0.81 \\
      \text{New position: } (0.59, -0.81)\]
e.  $-\pi$

\[
\begin{align*}
\cos(\theta - \pi) &= -\cos(\theta) = -0.81 \\
\sin(\theta - \pi) &= -\sin(\theta) = -0.59
\end{align*}
\]

New position: $(-0.81, -0.59)$

f.  $\frac{\pi}{2} - \theta$

\[
\begin{align*}
\cos\left(\theta + \left(\frac{\pi}{2} - \theta\right)\right) &= \cos\left(\frac{\pi}{2}\right) = 0 \\
\sin\left(\theta + \left(\frac{\pi}{2} - \theta\right)\right) &= \sin\left(\frac{\pi}{2}\right) = 1
\end{align*}
\]

New position: $(0, 1)$

g.  $\pi - 2\theta$

\[
\begin{align*}
\cos(\theta + (\pi - 2\theta)) &= \cos(\pi - \theta) = -\cos(\theta) = -0.81 \\
\sin(\theta + (\pi - 2\theta)) &= \sin(\pi - \theta) = -\sin(\theta) = 0.59
\end{align*}
\]

New position: $(-0.81, 0.59)$

h.  $-2\theta$

\[
\begin{align*}
\cos(\theta - 2\theta) &= \cos(-\theta) = \cos(\theta) = 0.81 \\
\sin(\theta - 2\theta) &= \sin(-\theta) = -\sin(\theta) = -0.59
\end{align*}
\]

New position: $(0.81, -0.59)$