Lesson 7: Curves from Geometry

Classwork

Exercise

Points $F$ and $G$ are located at $(0, 3)$ and $(0, -3)$. Let $P(x, y)$ be a point such that $PF + PG = 8$. Use this information to show that the equation of the ellipse is $\frac{x^2}{7} + \frac{y^2}{16} = 1$. 
Problem Set

1. Derive the equation of the ellipse with the given foci \( F \) and \( G \) that passes through point \( P \). Write your answer in standard form: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).
   
   a. The foci are \( F (-2,0) \) and \( G (2,0) \), and point \( P (x, y) \) satisfies the condition \( PF + PG = 5 \).
   
   b. The foci are \( F (-1,0) \) and \( G (1,0) \), and point \( P (x, y) \) satisfies the condition \( PF + PG = 5 \).
   
   c. The foci are \( F (0,-1) \) and \( G (0,1) \), and point \( P (x, y) \) satisfies the condition \( PF + PG = 5 \).
   
   d. The foci are \( F (-\frac{2}{3},0) \) and \( G (\frac{2}{3},0) \), and point \( P (x, y) \) satisfies the condition \( PF + PG = 3 \).
   
   e. The foci are \( F (0,-5) \) and \( G (0,5) \), and point \( P (x, y) \) satisfies the condition \( PF + PG = 12 \).
   
   f. The foci are \( F (-6,0) \) and \( G (6,0) \), and point \( P (x, y) \) satisfies the condition \( PF + PG = 20 \).

2. Recall from Lesson 6 that the semi-major axes of an ellipse are the segments from the center to the farthest vertices, and the semi-minor axes are the segments from the center to the closest vertices. For each of the ellipses in Problem 1, find the lengths \( a \) and \( b \) of the semi-major axes.

3. Summarize what you know about equations of ellipses centered at the origin with vertices \((a,0),(−a,0),(0,b),\) and \((0,−b)\).

4. Use your answer to Problem 3 to find the equation of the ellipse for each of the situations below.
   
   a. An ellipse centered at the origin with \( x \)-intercepts \((-2,0),(2,0)\) and \( y \)-intercepts \((0,8),(0,−8)\)
   
   b. An ellipse centered at the origin with \( x \)-intercepts \((-\sqrt{5},0),(\sqrt{5},0)\) and \( y \)-intercepts \((0,3),(0,−3)\)

5. Examine the ellipses and the equations of the ellipses you have worked with, and describe the ellipses with equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) in the three cases \( a > b, a = b, \) and \( b > a \).

6. Is it possible for \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \) to have foci at \((-c,0)\) and \((c,0)\) for some real number \( c \)?

7. For each value of \( k \) specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation \( \frac{x^2}{4} + y^2 = k \).
   
   a. \( k = 1 \)
   
   b. \( k = \frac{1}{4} \)
   
   c. \( k = \frac{1}{9} \)
   
   d. \( k = \frac{1}{16} \)
   
   e. \( k = \frac{1}{25} \)
f. \( k = \frac{1}{100} \)

g. Make a conjecture: Which points in the plane satisfy the equation \( \frac{x^2}{4} + y^2 = 0 \)?

h. Explain why your conjecture in part (g) makes sense algebraically.

i. Which points in the plane satisfy the equation \( \frac{x^2}{4} + y^2 = -1 \)?

8. For each value of \( k \) specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation \( \frac{x^2}{k} + y^2 = 1 \).
   a. \( k = 1 \)
   b. \( k = 2 \)
   c. \( k = 4 \)
   d. \( k = 10 \)
   e. \( k = 25 \)
   f. Describe what happens to the graph of \( \frac{x^2}{k} + y^2 = 1 \) as \( k \to \infty \).

9. For each value of \( k \) specified in parts (a)–(e), plot the set of points in the plane that satisfy the equation \( x^2 + \frac{y^2}{k} = 1 \).
   a. \( k = 1 \)
   b. \( k = 2 \)
   c. \( k = 4 \)
   d. \( k = 10 \)
   e. \( k = 25 \)
   f. Describe what happens to the graph of \( x^2 + \frac{y^2}{k} = 1 \) as \( k \to \infty \).