Lesson 16: Similar Triangles in Circle-Secant (or Circle-Secant-Tangent) Diagrams

Student Outcomes
- Students find missing lengths in circle-secant or circle-secant-tangent diagrams.

Lesson Notes
The Opening Exercise reviews Lesson 15, secant lines that intersect outside of circles. In this lesson, students continue the study of secant lines and circles, but the focus changes from angles formed to segment lengths and their relationships to each other. Exploratory Challenges 1 and 2 allow students to measure the segments formed by intersecting secant lines and develop their own formulas. Exploratory Challenge 3 has students prove the formulas that they developed in the first two Exploratory Challenges.

This lesson focuses heavily on MP.8, as students work to articulate relationships among segment lengths by noticing patterns in repeated measurements and calculations.

Classwork
Opening Exercise (5 minutes)
Several relationships between angles and arcs of a circle have just been studied. This exercise, which should be completed individually, asks students to state the type of angle and the angle/arc relationship and then find the measure of an arc. Use this as an informal assessment to monitor student understanding.

Opening Exercise
Identify the type of angle and the angle/arc relationship, and then find the measure of \( x \).

- \( x = 58 \); the inscribed angle is equal to half intercepted arc.
- \( x = 86 \); angle formed by secants intersecting inside the circle is half the sum of arcs intercepted by angle and its vertical angle.
Exploratory Challenge 1 (10 minutes)

In Exploratory Challenge 1, students study the relationships of segments of secant lines intersecting inside of circles. Students measure and then find a formula. Allow students to work in pairs, and have them construct more circles with secants crossing at exterior points until they see the relationship. Students need a ruler.

If chords of a circle intersect, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord; \( a \cdot b = c \cdot d \).

Note that the actual measurements are not included due to the difference in the electronic form vs. paper form of the images. Have the measurements completed as part of lesson preparation.

Exploratory Challenge 1

Measure the lengths of the chords in centimeters, and record them in the table.

<table>
<thead>
<tr>
<th>A.</th>
<th>B.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
</tr>
</tbody>
</table>
What relationship did you discover?

- $a \cdot b = c \cdot d$

Say that to your neighbor in words.

- If chords of a circle intersect, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.

**Exploratory Challenge 2 (10 minutes)**

In the second Exploratory Challenge, the point of intersection is outside of the circle, and students try to develop an equation that works. Students should continue this work in groups.

**Table:**

<table>
<thead>
<tr>
<th>Circle</th>
<th>$a$ (cm)</th>
<th>$b$ (cm)</th>
<th>$c$ (cm)</th>
<th>$d$ (cm)</th>
<th>Do you notice a relationship?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>All are the same measure.</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Not sure</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$a \cdot b = c \cdot d$</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$a \cdot b = c \cdot d$</td>
</tr>
</tbody>
</table>
Exploratory Challenge 2
Measure the lengths of the chords in centimeters, and record them in the table.

<table>
<thead>
<tr>
<th>Circle</th>
<th>a (cm)</th>
<th>b (cm)</th>
<th>c (cm)</th>
<th>d (cm)</th>
<th>Do you notice a relationship?</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Product of (a) and (b) is equal to product of (c) and (d).</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Products are not equal for these measurements.</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(a(a + b) = c(c + d))</td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(a(a + b) = c(c + d))</td>
</tr>
</tbody>
</table>

- Does the same relationship hold?
  - No
- Did you discover a different relationship?
  - Yes, \(a(a + b) = c(c + d)\).
- Explain the two relationships that you just discovered to your neighbor and when to use each formula.
  - When secant lines intersect inside a circle, use \(a \cdot b = c \cdot d\).
  - When secant lines intersect outside of a circle, use \(a(a + b) = c(c + d)\).
Exploratory Challenge 3 (12 minutes)

Students have just discovered relationships between the segments of secant and tangent lines and circles. In Exploratory Challenge 3, they prove why the formulas work mathematically.

Display the diagram to the right on the board.

- We are going to prove mathematically why the formulas we found in Exploratory Challenges 1 and 2 are valid using similar triangles.
- Draw $BD$ and $EC$.
- Take a few minutes with a partner and prove that $\triangle BFD$ is similar to $\triangle EFC$.

Allow students time to work while circulating around the room. Help groups that are struggling. Bring the class back together, and have students share their proofs.

- $m\angle BFD = m\angle EFC$  
  *Vertical angles*
- $m\angle BDF = m\angle ECF$  
  *Inscribed in same arc*
- $m\angle DBF = m\angle ECF$  
  *Inscribed in same arc*
- $\triangle BFD \sim \triangle EFC$  
  *AA*

What is true about similar triangles?

- *Corresponding sides are proportional.*

Write a proportion involving sides $BF$, $FC$, $DF$, and $FE$.

- $\frac{BF}{FE} = \frac{DF}{FC}$

Can you rearrange this to prove the formula discovered in Exploratory Challenge 1?

- $(BF)(FC) = (DF)(FE)$

Display the next diagram on the board.

- Now, let’s try to prove the formula we found in Exploratory Challenge 2.
- Name two triangles that could be similar.
  - $\triangle CFB$ and $\triangle CED$  
- Take a few minutes with a partner and prove that $\triangle CFB$ is similar to $\triangle CED$.

Allow students time to work while circulating around the room. Help groups that are struggling. Bring the class back together, and have students share their proofs.

- $m\angle C = m\angle C$  
  *Common angle*
- $m\angle CBF = m\angle CDE$  
  *Inscribed in same arc*
- $\triangle CFB \sim \triangle CED$  
  *AA*

Write a proportion that will be true.

- $\frac{CB}{CD} = \frac{CF}{CE}$
• Can you rearrange this to prove the formula discovered in Exploratory Challenge 2?
  - \((CE)(CB) = (CF)(CD)\)
• What if one of the lines is tangent and the other is secant? (Show diagram.)

Students should be able to reason that \(a \cdot a = b(b + c)\)
- \(a^2 = b(b + c)\)
- \(a = \sqrt{b(b + c)}\)

Closing (3 minutes)

We have just concluded our study of secant lines, tangent lines, and circles. In Lesson 15, you completed a table about angle relationships. This summary completes the table adding segment relationships. Complete the table below, and compare your answers with your neighbor. Bring the class back together to discuss answers to ensure students have the correct formulas in their tables.

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### The Inscribed Angle Theorem and Its Family

<table>
<thead>
<tr>
<th>Diagram</th>
<th>How the two shapes overlap</th>
<th>Relationship between (a, b, c, \text{ and } d)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Two secant lines intersecting in the interior of the circle." /></td>
<td>Two secant lines intersecting in the interior of the circle.</td>
<td>(a \cdot b = c \cdot d)</td>
</tr>
<tr>
<td><img src="image" alt="Two secant lines intersecting in the exterior of the circle." /></td>
<td>Two secant lines intersecting in the exterior of the circle.</td>
<td>(a(a + b) = c(c + d))</td>
</tr>
</tbody>
</table>
Lesson Summary

**THEOREMS:**

- When secant lines intersect inside a circle, use $a \cdot b = c \cdot d$.
- When secant lines intersect outside of a circle, use $a(a + b) = c(c + d)$.
- When a tangent line and a secant line intersect outside of a circle, use $a^2 = b(b + c)$.

**Relevant Vocabulary**

**SECANT TO A CIRCLE:** A secant line to a circle is a line that intersects a circle in exactly two points.

Exit Ticket (5 minutes)
Lesson 16: Similar Triangles in Circle-Secant (or Circle-Secant-Tangent) Diagrams

Exit Ticket

In the circle below, \( m \angle GF = 30^\circ \), \( m \angle DE = 120^\circ \), \( CG = 6 \), \( GH = 2 \), \( FH = 3 \), \( CF = 4 \), \( HE = 9 \), and \( FE = 12 \).

a. Find \( a \) (\( m \angle DHE \)).

b. Find \( b \) (\( m \angle DCE \)), and explain your answer.

c. Find \( x \) (\( HD \)), and explain your answer.

d. Find \( y \) (\( DG \)).
Exit Ticket Sample Solutions

In the circle below, \( m\overset{\frown}{GF} = 30^\circ \), \( m\overset{\frown}{BE} = 120^\circ \), \( CG = 6 \), \( GH = 2 \), \( FH = 3 \), \( CF = 4 \), \( HE = 9 \), and \( FE = 12 \).

a. Find \( \alpha \) (\( m\angle DHE \)).
   \[ \alpha = 75^\circ \]

b. Find \( \beta \) (\( m\angle DCE \)), and explain your answer.
   \[ \beta = 45^\circ \]; \( \beta \) is an angle with its vertex outside of the circle, so it has a measure half the difference between its larger and smaller intercepted arcs.

c. Find \( x \) (\( HD \)), and explain your answer.
   \[ x = 6 \]; \( x \) is part of a secant line intersecting another secant line inside the circle, so \( 2 \cdot 9 = 3 \cdot x \).

d. Find \( y \) (\( DG \)).
   \[ y = \frac{14}{3} = 4 \frac{2}{3} \]

Problem Set Sample Solutions

1. Find \( x \).
   \[ x = 8 \]

2. Find \( x \).
   \[ x(x + 1) = 2(2 + 4) \]
   \[ x^2 + x - 12 = 0 \]
   \[ (x + 4)(x - 3) = 0 \]
   \[ x = 3 \]
3. $DF < FB, DF \neq 1, DF < FE,$ and all values are integers; prove $DF = 3.$

$7 \cdot 6 = 42,$ so $DF \cdot FE$ must equal 42. If $DF < FE,$ $DF$ could equal 1, 3, or 6. $DF \neq 1$ and $DF < FB,$ so $DF$ must equal 3.

4. $CE = 6, CB = 9,$ and $CD = 18.$ Show $CF = 3.$

$6 \cdot 9 = 54$ and $18 \cdot CF = 54.$ This means $CF = 3.$

5. Find $x.$

$x = 2\sqrt{13}$

6. Find $x.$

$x = 11.25$

7. Find $x.$

$(x - 27)x = 8(20)$

$x = 32$

8. Find $x.$

$x(x + 7) = (x + 3)^2$

$x = 9$
9. In the circle shown, \( DE = 11, BC = 10, \) and \( DF = 8. \) Find \( FE, BF, FC. \)

\[
x(10 - x) = (3)(8)
\]

\[
FE = 3, BF = 4, FC = 6
\]

10. In the circle shown, \( m \angle DBG = 150^\circ, m \angle DB = 30^\circ, m \angle CEF = 60^\circ, DF = 8, DB = 4, \) and \( GF = 12. \)

   a. Find \( m \angle GDB. \)

\[
60^\circ
\]

   b. Prove \( \triangle DBF \sim \triangle ECF \)

   \[
m \angle DBF = m \angle CEF
\]

Inscribed angle \( \angle DBF \) is half the measure of intercepted arc, \( \overarc{DC} \) and \( \angle CEF \) formed by a tangent line and a secant line is also half the measure of the same intercepted arc \( \overarc{DC}. \)

   \[
m \angle DFB = m \angle EFC
\]

Vertical angles are equal in measure.

\( \triangle DBF \sim \triangle ECF \)  AA

   c. Set up a proportion using \( \overline{CE} \) and \( \overline{GE}. \)

\[
\frac{8}{GE + 12} = \frac{4}{CE} \quad \text{or} \quad 2CE = GE + 12
\]

   d. Set up an equation with \( CE \) and \( GE \) using a theorem for segment lengths from this section.

\[
CE^2 = GE(GE + 20)
\]