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# Rational and Exponential Functions

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<sup>1</sup>Each lesson is ONE day, and ONE day is considered a 45-minute period.

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*Topics A through C (assessment 1 day; return, remediation, or further applications 2 days)*

## Precalculus and Advanced Topics • Module 3

**Rational and Exponential Functions****OVERVIEW**

Students encountered the fundamental theorem of algebra, that every polynomial function has at least one zero in the realm of the complex numbers (**N-CN.C.9**), in Algebra II Module 1. Topic A of this module brings students back to the study of complex roots of polynomial functions. Students first briefly review quadratic and cubic functions and then extend familiar polynomial identities to both complex numbers (**N-CN.C.8**) and to general polynomial functions. Students use polynomial identities to find square roots of complex numbers. The binomial theorem and its relationship to Pascal's triangle are explored using roots of unity (**A-APR.C.5**). Topic A concludes with students' use of Cavalieri's principle to derive formulas (**G-GPE.A.3**) for the volume of the sphere and other geometric solids (**G-GMD.A.2**).

In Topic B, students explore composition of functions in depth (**F-BF.A.1c**) and notice that a composition of a polynomial function with the function  $f(x) = \frac{1}{x}$  gives functions that can be written as ratios of polynomial functions. A study of rational expressions shows that these expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression (**A-APR.D.7**). Focus then turns to restricted domain values as students compare the properties of rational functions represented in different ways (**F-IF.C.9**). Students apply these operations to simplify rational expressions and go on to graphing rational functions, identifying zeros and asymptotes, and analyzing end behavior (**F-IF.C.7d**).

The module ends with Topic C, in which students study inverse functions, being careful to understand when inverse functions do and do not exist, working to restrict the domain of a function to produce an invertible function. They compare and create different representations of functions including tables and graphs. Students compose functions to verify that one function is the inverse of another and work with tables of data to identify features of inverse functions (**F-BF.B.4b**, **F-BF.B.4c**, **F-BF.B.4d**). Special emphasis is given to the inverse relationship between exponential and logarithmic functions (**F-BF.B.5**).

**Focus Standards****Use complex numbers in polynomial identities and equations.**

- N-CN.C.8** (+) Extend polynomial identities to the complex numbers. *For example, rewrite  $x^2 + 4$  as  $(x + 2i)(x - 2i)$ .*
- N-CN.C.9** (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

**Use polynomial identities to solve problems.**

- A-APR.C.5** (+) Know and apply the Binomial Theorem for the expansion of  $(x + y)^n$  in powers of  $x$  and  $y$  for a positive integer  $n$ , where  $x$  and  $y$  are any numbers, with coefficients determined for example by Pascal's Triangle.<sup>2</sup>

**Rewrite rational expressions.**

- A-APR.D.7** (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

**Analyze functions using different representations.**

- F-IF.C.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.\*
- d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior
- F-IF.C.9** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*<sup>3</sup>

**Build a function that models a relationship between two quantities.**

- F-BF.A.1** Write a function that describes a relationship between two quantities.\*
- c. (+) Compose functions. *For example, if  $T(y)$  is the temperature in the atmosphere as a function of height, and  $h(t)$  is the height of a weather balloon as a function of time, then  $T(h(t))$  is the temperature at the location of the weather balloon as a function of time.*

**Build new functions from existing functions.**

- F-BF.B.4** Find inverse functions.
- b. (+) Verify by composition that one function is the inverse of another.
- c. (+) Read value of an inverse function from a graph or a table, given that the function has an inverse.
- d. (+) Produce an invertible function from a non-invertible function by restricting the domain.
- F-BF.B.5** (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

<sup>2</sup>The Binomial Theorem can be proved by mathematical induction or by a combinatorial argument.

<sup>3</sup>This standard is to be applied to rational functions.

**Explain volume formulas and use them to solve problems.**

**G-GMD.A.2** (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

**Translate between the geometric description and the equation for a conic section.**

**G-GPE.A.3** (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

## Foundational Standards

**Reason quantitatively and use units to solve problems.**

**N-Q.A.2** Define appropriate quantities for the purpose of descriptive modeling.\*

**Perform arithmetic operations with complex numbers.**

**N-CN.A.1** Know there is a complex number  $i$  such that  $i^2 = -1$ , and every complex number has the form  $a + bi$  with  $a$  and  $b$  real.

**N-CN.A.2** Use the relation  $i^2 = -1$  and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

**Use complex numbers in polynomial identities and equations.**

**N-CN.C.7** Solve quadratic equations with real coefficients that have complex solutions.

**N-CN.C.8** (+) Extend polynomial identities to the complex numbers. *For example, rewrite  $x^2 + 4$  as  $(x + 2i)(x - 2i)$ .*

**Interpret the structure of expressions.**

**A-SSE.A.1** Interpret expressions that represent a quantity in terms of its context.\*

- Interpret parts of an expression, such as terms, factors, and coefficients.
- Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret  $P(1 + r)^n$  as the product of  $P$  and a factor not depending on  $P$ .*

**Write expressions in equivalent forms to solve problems.**

**A-SSE.B.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.\*

- Factor a quadratic expression to reveal the zeros of the function it defines.

- b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
- c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression  $1.15^t$  can be rewritten as  $(1.15^{1/12})^{12t} \approx 1.012^{12t}$  to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

### Create equations that describe numbers or relationships.\*

- A-CED.A.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*
- A-CED.A.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
- A-CED.A.3** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*
- A-CED.A.4** Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law  $V = IR$  to highlight resistance  $R$ .*

### Understand solving equations as a process of reasoning and explain the reasoning.

- A-REI.A.1** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

### Solve equations and inequalities in one variable.

- A-REI.B.3** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

### Solve systems of equations.

- A-REI.C.6** Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

### Experiment with transformations in the plane.

- G-CO.A.2** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
- G-CO.A.4** Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

- G-CO.A.5** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

### Extend the domain of trigonometric functions using the unit circle.

- F-TF.A.1** Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
- F-TF.A.2** Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
- F-TF.A.3** (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for  $\pi/3$ ,  $\pi/4$ , and  $\pi/6$ , and use the unit circle to express the values of sine, cosine, and tangent for  $\pi - x$ ,  $\pi + x$ , and  $2\pi - x$  in terms of their values for  $x$ , where  $x$  is any real number.

### Prove and apply trigonometric identities.

- F-TF.C.8** Prove the Pythagorean identity  $\sin^2(\theta) + \cos^2(\theta) = 1$  and use it to find  $\sin(\theta)$ ,  $\cos(\theta)$ , or  $\tan(\theta)$  given  $\sin(\theta)$ ,  $\cos(\theta)$ , or  $\tan(\theta)$  and the quadrant of the angle.

## Focus Standards for Mathematical Practice

- MP.3** **Construct viable arguments and critique the reasoning of others.** Students construct arguments and critique the reasoning of others when making conjectures about the roots of polynomials (a polynomial of degree 3 will have three roots) and solving problems by applying algebraic properties. Students determine the domain and range of rational functions and reason what effect these restrictions have on the graph of the rational function. Students use reasoning to argue that restricting the domain of a function allows for the construction of an inverse function.
- MP.7** **Look for and make use of structure.** Students use polynomial identities to determine roots of polynomials and square roots of complex numbers. They relate the structure of rational expressions to the graphs of rational functions by studying transformations of these graphs. Students determine the relationship between functions and their inverses.
- MP.8** **Look for and express regularity in repeated reasoning.** Students use prior knowledge of the fundamental theorem of algebra to justify the number of roots of unity. Students develop understanding of the binomial theorem through repeated binomial expansions and connecting their observations to patterns in Pascal's triangle. In performing and reasoning about several computations with fractional expressions, students extend the properties of rational numbers to rational expressions.

## Terminology

### New or Recently Introduced Terms

- **Ellipse** (An *ellipse* is the set of all points in a plane such that the sum of the distances from two points (foci) to any point on the line is constant. Given  $k$ , foci  $A$  and  $B$ , and any point  $P$  on the ellipse,  $PA + PB = k$ .)

Standard equation for an ellipse:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

An ellipse whose foci points are the same point, that is,  $A = B$ , is a circle.

- **Horizontal Asymptote** (Let  $L$  be a real number. The line given by the equation  $y = L$  is a *horizontal asymptote* of the graph of  $y = f(x)$  if at least one of the following statements is true.
  - As  $x$  approaches infinity,  $f(x)$  approaches  $L$ .
  - As  $x$  approaches negative infinity,  $f(x)$  approaches  $L$ .)

- **Hyperbola** (A *hyperbola* is the set of points in a plane whose distances to two fixed points  $A$  and  $B$ , called the foci, have a constant difference. Given  $P$  and a positive constant,  $k$ ,  $|PA - PB| = k$ .)

Standard equation for a hyperbola:  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  or  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

- **Vertical Asymptote** (Let  $a$  be a real number. The line given by the equation  $x = a$  is a *vertical asymptote* of the graph of  $y = f(x)$  if at least one of the following statements is true.
  - As  $x$  approaches  $a$ ,  $f(x)$  approaches infinity.
  - As  $x$  approaches  $a$ ,  $f(x)$  approaches negative infinity.)

### Familiar Terms and Symbols<sup>4</sup>

- Complex Numbers
- Domain
- Exponential Function
- Inverse Functions
- Logarithmic Function
- Polar Form
- Sphere
- Transformation
- Volume

<sup>4</sup>These are terms and symbols students have seen previously.

## Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software
- Geogebra Software

## Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic A	Constructed response with rubric	N-CN.C.8, N-CN.C.9, A-APR.C.5, G-GMD.A.2, G-GPE.A.3
End-of-Module Assessment Task	After Topic C	Constructed response with rubric	A-APR.C.7, F-IF.C.7d, F-IF.C.9, F-BF.A.1c, F-BF.B.4b, F-BF.B.4c, F-BF.B.4d, F-BF.B.5