Lesson 32: Using Trigonometry to Find Side Lengths of an Acute Triangle

Student Outcomes

- Students find missing side lengths of an acute triangle given one side length and the measures of two angles.
- Students find the missing side length of an acute triangle given two side lengths and the measure of the included angle.

Lesson Notes

In Lesson 32, students learn how to determine unknown lengths in acute triangles. Once again, they drop an altitude in the given triangle to create right triangles and use trigonometric ratios and the Pythagorean theorem to solve triangle problems (G-SRT.C.8). This lesson and the next introduce the law of sines and cosines (G-SRT.D.10 and G-SRT.D.11).

Based on availability of time in the module, teachers may want to divide the lesson into two parts by addressing everything until Exercises 1–2 on one day and the remaining content on the following day.

Classwork

Opening Exercise (3 minutes)

The objective for part (b) is that students realize that $x$ and $y$ cannot be found using the method they know with trigonometric ratios.

Opening Exercise

a. Find the lengths of $d$ and $e$.

\[
\sin 60 = \frac{5}{e}; e = \frac{10}{\sqrt{3}}
\]

\[
\cos 60 = \frac{d}{10}; d = \frac{5}{\sqrt{3}}
\]

b. Find the lengths of $x$ and $y$. How is this different from part (a)?

Accept any reasonable answer explaining that the triangle is not a right triangle; therefore, the trigonometric ratios used in part (a) are not applicable here.
Discussion (10 minutes)

Lead students through an explanation of the law of sines for acute triangles.

- Today we will show how two facts in trigonometry aid us to find unknown measurements in triangles that are not right triangles; we can use these facts for acute and obtuse triangles, but today we will specifically study acute triangles.

The facts are called the law of sines and the law of cosines. We begin with the law of sines.

- **LAW OF SINES**: For an acute triangle \( \triangle ABC \) with \( \angle A \), \( \angle B \), and \( \angle C \) and the sides opposite them \( a \), \( b \), and \( c \), the law of sines states:

\[
\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c}
\]

- Restate the law of sines in your own words with a partner.

This may be difficult for students to articulate generally, without reference to a specific angle. State it for students if they are unable to state it generally.

  - The ratio of the sine of an angle in a triangle to the side opposite the angle is the same for each angle in the triangle.

- Consider \( \triangle ABC \) with an altitude drawn from \( B \) to \( \overline{AC} \). What is \( \sin \angle C \)?

  - \( \sin \angle C = \frac{h}{a} \)

- Therefore, \( h = a \sin \angle C \).

- What is \( \sin A \)?

  - \( \sin \angle A = \frac{h}{c} \)

- Therefore, \( h = c \sin \angle A \).

- What can we conclude so far?

  - Since \( h = a \sin \angle C \) and \( h = c \sin \angle A \), then \( a \sin \angle C = c \sin \angle A \).

- With a little algebraic manipulation, we can rewrite \( a \sin \angle C = c \sin \angle A \) as \( \frac{\sin \angle A}{a} = \frac{\sin \angle C}{c} \).

- We have partially shown why the law of sines is true. What do we need to show in order to complete the proof, and how can we go about determining this?

Allow students a few moments to try and develop this argument independently.
We need to show that \( \frac{\sin \angle B}{b} \) is equal to \( \frac{\sin \angle A}{a} \) and to \( \frac{\sin \angle C}{c} \). If we draw a different altitude, we can achieve this drawing:

\[
\begin{aligned}
\text{An altitude from } A \text{ gives us } \\
\sin \angle B &= \frac{h}{c} \quad \text{or } h = c \sin \angle B.
\end{aligned}
\]

Also, \( \sin \angle C = \frac{h}{b} \) or \( h = b \sin \angle C \). Therefore, \( c \sin \angle B = b \sin \angle C \) or \( \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c} \).

Since \( \frac{\sin \angle A}{a} = \frac{\sin \angle C}{c} \) and \( \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c} \), then \( \frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c} \).

As soon as it has been established that \( \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c} \), the proof is really done, as the angles selected are arbitrary and, therefore, apply to any angle within the triangle. This can be explained for students if they are ready for the explanation.

**Example 1 (4 minutes)**

Students apply the law of sines to determine unknown measurements within a triangle.

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**Example 1**

A surveyor needs to determine the distance between two points \( A \) and \( B \) that lie on opposite banks of a river. A point \( C \) is chosen 160 meters from point \( A \), on the same side of the river as \( A \). The measures of \( \angle BAC \) and \( \angle ACB \) are 41° and 55°, respectively. Approximate the distance from \( A \) to \( B \) to the nearest meter.

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Allow students a few moments to begin the problem before assisting them.
What measurement can we add to the diagram based on the provided information?

- The measurement of \( \angle B \) must be 84° by the triangle sum theorem.

Use the law of sines to set up all possible ratios applicable to the diagram.

- \( \frac{\sin 41}{a} = \frac{\sin 84}{160} = \frac{\sin 55}{c} \)

Which ratios will be relevant to determining the distance from \( A \) to \( B \)?

- \( \frac{\sin 84}{160} = \frac{\sin 55}{c} \)

Solve for \( c \).

- \( c = \frac{160 \sin 55}{\sin 84} \)
- \( c = 132 \)

The distance from \( A \) to \( B \) is 132 m.

Exercises 1–2 (6 minutes)

Depending on the time available, consider having students move directly to Exercise 2.

### Exercises 1–2

1. In \( \triangle ABC \), \( m \angle A = 30^\circ \), \( a = 12 \), and \( b = 10 \). Find \( \sin \angle B \). Include a diagram in your answer.

\[
\frac{\sin 30}{12} = \frac{\sin \angle B}{10} \\
\sin \angle B = \frac{5}{12}
\]

2. A car is moving toward a tunnel carved out of the base of a hill. As the accompanying diagram shows, the top of the hill, \( H \), is sighted from two locations, \( A \) and \( B \). The distance between \( A \) and \( B \) is 250 ft. What is the height, \( h \), of the hill to the nearest foot?

Let \( x \) represent \( BH \), in feet. Applying the law of sines,

\[
\frac{\sin 15}{250} = \frac{\sin 30}{x} \\
x = \frac{250 \sin 30}{\sin 15} \\
x = \frac{125}{\sin 15} \\
x \approx 482.96
\]
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**Discussion (10 minutes)**

Lead students through an explanation of the law of cosines for acute triangles. If more time is needed to cover the following Discussion, skip Exercise 1 as suggested earlier to allow for more time here.

- The next fact we will examine is the law of cosines.
- **Law of Cosines:** For an acute triangle $\triangle ABC$ with $\angle A$, $\angle B$, and $\angle C$ and the sides opposite them $a$, $b$, and $c$, the law of cosines states
  \[ c^2 = a^2 + b^2 - 2ab \cos \angle C. \]

- State the law of cosines in your own words.
  - The square of one side of the triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the angle between them.

The objective of being able to state the law in words is to move the focus away from specific letters and generalize the formula for any situation.

- The law of cosines is a generalization of the Pythagorean theorem, which can only be used with right triangles.
- Substitute $90^\circ$ for $m\angle C$ into the law of cosines formula, and observe the result.
  - $c^2 = a^2 + b^2 - 2ab \cos 90$
- What is the value of $\cos 90^\circ$? What happens to the equation?
  - $\cos 90^\circ = 0$; the equation simplifies to $c^2 = a^2 + b^2$.
- This explains why the law of cosines is a generalization of the Pythagorean theorem. We use the law of cosines for acute triangles in order to determine the side length of the third side of a triangle, provided two side lengths and the included angle measure.

**Scaffolding:**

For students ready for a challenge, instead of asking them to substitute $90^\circ$ for $\angle C$, ask them what case results with the law of cosines being reduced to the Pythagorean theorem.
We return to \( \triangle ABC \) from Example 1, with an altitude drawn from \( B \) to \( AC \), but this time the point where the altitude meets \( AC \), point \( D \), divides \( AC \) into lengths \( d \) and \( e \).

Express \( e \) and \( h \) using trigonometry with respect to \( \angle C \).
- \( h = a \sin \angle C \)
- \( e = a \cos \angle C \)

Now we turn to right triangle \( \triangle ABD \). What length relationship can be concluded between the sides of the triangle?
- By the Pythagorean theorem, the length relationship in \( \triangle ABD \) is \( c^2 = d^2 + h^2 \).

Substitute the trigonometric expressions for \( d \) and \( h \) into this statement. Notice you will need length \( b \).
- Then, the statement becomes \( c^2 = (b - a \cos \angle C)^2 + (a \sin \angle C)^2 \).

Simplify this statement as much as possible.
- The statement becomes:
  \[
  c^2 = b^2 - 2ab \cos \angle C + a^2 \cos^2 \angle C + a^2 \sin^2 \angle C
  
  c^2 = b^2 - 2ab \cos \angle C + a^2((\cos \angle C)^2 + (\sin \angle C)^2)
  
  c^2 = b^2 - 2ab \cos \angle C + a^2(1)
  
  c^2 = a^2 + b^2 - 2ab \cos \angle C
  
\]

Notice that the right-hand side is composed of two side lengths and the cosine of the included angle. This will remain true if the labeling of the triangle is rearranged.

What are all possible arrangements of the law of cosines for \( \triangle ABC \)?
- \( a^2 = b^2 + c^2 - 2(bc) \cos \angle A \)
- \( b^2 = a^2 + c^2 - 2(ac) \cos \angle B \)
Example 2 (4 minutes)

Our friend the surveyor from Example 1 is doing some further work. He has already found the distance between points \(A\) and \(B\) (from Example 1). Now he wants to locate a point \(D\) that is equidistant from both \(A\) and \(B\) and on the same side of the river as \(A\). He has his assistant mark the point \(D\) so that \(\angle ABD\) and \(\angle BAD\) both measure 75°. What is the distance between \(D\) and \(A\) to the nearest meter?

- What do you notice about \(\triangle ABD\) right away?
  - \(\triangle ABD\) must be an isosceles triangle since it has two angles of equal measure.
- We must keep this in mind going forward. Add all relevant labels to the diagram.
  - Students should add the distance of 132 m between \(A\) and \(B\) and add the label of \(a\) and \(b\) to the appropriate sides.
- Set up an equation using the law of cosines. Remember, we are trying to find the distance between \(D\) and \(A\) or, as we have labeled it, \(b\).
  - \(b^2 = 132^2 + a^2 - 2(132)(a) \cos 75\)
- Recall that this is an isosceles triangle; we know that \(a = b\). To reduce variables, we will substitute \(b\) for \(a\). Rewrite the equation, and solve for \(b\).
  - Sample solution:
    
    \[
    b^2 = 132^2 + (b)^2 - 2(132)(b) \cos 75
    \]
    \[
    b^2 = 132^2 + (b)^2 - 264(b) \cos 75
    \]
    \[
    0 = 132^2 - 264(b) \cos 75
    \]
    \[
    264(b) \cos 75 = 132^2
    \]
    \[
    b = \frac{132^2}{264 \cos 75}
    \]
    \[
    b \approx 255 \text{ m}
    \]
Exercise 3 (2 minutes)

3. Parallelogram $ABCD$ has sides of lengths 44 mm and 26 mm, and one of the angles has a measure of $100^\circ$. Approximate the length of diagonal $AC$ to the nearest millimeter.

In parallelogram $ABCD$, $m \angle C = 100^\circ$; therefore, $m \angle D = 80^\circ$.

Let $d$ represent the length of $AC$.

$$d^2 = 44^2 + 26^2 - 2(44)(26) \cos 80^\circ$$

$$d = 47$$

The length of $AC$ is 47 millimeters.

Closing (1 minute)

Ask students to summarize the key points of the lesson. Additionally, consider asking students to answer the following questions independently in writing, to a partner, or to the whole class.

- In what kinds of cases are we applying the laws of sines and cosines?
  - We apply the laws of sines and cosines when we do not have right triangles to work with. We used the laws of sines and cosines for acute triangles.

- State the law of sines. State the law of cosines.
  - For an acute triangle $\triangle ABC$ with $\angle A$, $\angle B$, and $\angle C$ and the sides opposite them $a$, $b$, and $c$,
    - The law of cosines states: $c^2 = a^2 + b^2 - 2ab \cos \angle C$.
    - The law of sines states: $\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin \angle C}{c}$.

Exit Ticket (5 minutes)
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1. Use the law of sines to find lengths $b$ and $c$ in the triangle below. Round answers to the nearest tenth as necessary.

2. Given $\triangle DEF$, use the law of cosines to find the length of the side marked $d$ to the nearest tenth.
Exit Ticket Sample Solutions

1. Use the law of sines to find lengths $b$ and $c$ in the triangle below. Round answers to the nearest tenth as necessary.

   $m \angle C = 82^\circ$

   $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

   $\frac{\sin 42}{18} = \frac{\sin 56}{b} = \frac{\sin 82}{c}$

   $b = \frac{18(\sin 56)}{\sin 42} \approx 22.3$

   $c = \frac{18(\sin 82)}{\sin 42} \approx 26.6$

2. Given $\triangle DEF$, use the law of cosines to find the length of the side marked $d$ to the nearest tenth.

   $d^2 = 6^2 + 9^2 - 2(6)(9)(\cos 65)$

   $d^2 = 36 + 81 - 108(\cos 65)$

   $d^2 = 117 - 108(\cos 65)$

   $d = \sqrt{117 - 108(\cos 65)}$

   $d \approx 8.4$

Problem Set Sample Solutions

1. Given $\triangle ABC$, $AB = 14$, $m \angle A = 57.2^\circ$, and $m \angle C = 78.4^\circ$, calculate the measure of angle $B$ to the nearest tenth of a degree, and use the law of sines to find the lengths of $AC$ and $BC$ to the nearest tenth.

   By the angle sum of a triangle, $m \angle B = 44.4^\circ$.

   $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

   $\frac{\sin 57.2}{a} = \frac{\sin 44.4}{b} = \frac{\sin 78.4}{14}$

   $a = \frac{14 \sin 57.2}{\sin 78.4} \approx 12.0$

   $b = \frac{14 \sin 44.4}{\sin 78.4} \approx 10.0$
Calculate the area of \( \triangle ABC \) to the nearest square unit.
\[
\text{Area} = \frac{1}{2} bc \sin A
\]
\[
\text{Area} = \frac{1}{2} (10)(14) \sin 57.2
\]
\[
\text{Area} = 70 \sin 57.2
\]
\[
\text{Area} \approx 59
\]

2. Given \( \triangle DEF \), \( \angle F = 39^\circ \), and \( EF = 13 \), calculate the measure of \( \angle E \), and use the law of sines to find the lengths of \( DF \) and \( DE \) to the nearest hundredth.

By the angle sum of a triangle, \( \angle E = 55^\circ \).

\[
\sin D = \sin E = \sin F
\]
\[
\frac{\sin D}{d} = \frac{\sin E}{e} = \frac{\sin F}{f}
\]
\[
\frac{\sin 86}{13} = \frac{\sin 55}{e} = \frac{\sin 39}{f}
\]
\[
e = \frac{13 \sin 55}{\sin 86} \\
e \approx 10.67
\]

3. Does the law of sines apply to a right triangle? Based on \( \triangle ABC \), the following ratios were set up according to the law of sines.

\[
\frac{\sin \angle A}{a} = \frac{\sin \angle B}{b} = \frac{\sin 90}{c}
\]

Fill in the partially completed work below:
\[
\frac{\sin \angle A}{a} = \frac{\sin 90}{c} \\
\frac{\sin \angle A}{a} = \frac{1}{c} \\
\sin \angle A = \frac{a}{c}
\]
\[
\frac{\sin \angle B}{b} = \frac{\sin 90}{c} \\
\frac{\sin \angle B}{b} = \frac{1}{c} \\
\sin \angle B = \frac{b}{c}
\]

What conclusions can we draw?

*The law of sines does apply to a right triangle. We get the formulas that are equivalent to \( \sin \angle A = \frac{\text{opp}}{\text{hyp}} \) and \( \sin \angle B = \frac{\text{opp}}{\text{hyp}} \), where \( A \) and \( B \) are the measures of the acute angles of the right triangle.*
4. Given quadrilateral $GHJK$, $\angle H = 50^\circ$, $\angle HKG = 80^\circ$, $\angle KGJ = 50^\circ$, $\angle J$ is a right angle, and $GH = 9$ in., use the law of sines to find the length of $GK$, and then find the lengths of $GJ$ and $JK$ to the nearest tenth of an inch.

*By the angle sum of a triangle, $\angle HKG = 50^\circ$; therefore, $\triangle GHK$ is an isosceles triangle since its base $\angle G$'s have equal measure.*

\[
\begin{align*}
\sin 50^\circ &= \frac{9 \sin 50^\circ}{h} \\
9 &= \frac{h \sin 50^\circ}{\sin 80^\circ} \\
h &\approx 7.0
\end{align*}
\]

$k = 7 \cos 50^\circ \approx 4.5$

$g = 7 \sin 50^\circ \approx 5.4$

5. Given triangle $LMN$, $LM = 10$, $LN = 15$, and $\angle L = 38^\circ$, use the law of cosines to find the length of $MN$ to the nearest tenth.

\[
\begin{align*}
l^2 &= 10^2 + 15^2 - 2(10)(15) \cos 38^\circ \\
l^2 &= 100 + 225 - 300 \cos 38^\circ \\
l^2 &= 325 - 300 \cos 38^\circ \\
l &= \sqrt{325 - 300 \cos 38^\circ} \\
l &\approx 9.4
\end{align*}
\]

$MN = 9.4$

*The length of $MN$ is approximately 9.4 units.*

6. Given triangle $ABC$, $AC = 6$, $AB = 8$, and $\angle A = 78^\circ$, draw a diagram of triangle $ABC$, and use the law of cosines to find the length of $BC$.

\[
\begin{align*}
a^2 &= 6^2 + 8^2 - 2(6)(8) \cos 78^\circ \\
a^2 &= 36 + 64 - 96 \cos 78^\circ \\
a^2 &= 100 - 96 \cos 78^\circ \\
a &= \sqrt{100 - 96 \cos 78^\circ} \\
a &\approx 8.9
\end{align*}
\]

*The length of $BC$ is approximately 8.9 units.*

Calculate the area of triangle $ABC$.

\[
\begin{align*}
\text{Area} &= \frac{1}{2}bc \sin A \\
\text{Area} &= \frac{1}{2}(6)(8) \sin 78^\circ \\
\text{Area} &= 23.5 \sin 78^\circ \\
\text{Area} &\approx 23.5
\end{align*}
\]

*The area of triangle $ABC$ is approximately 23.5 square units.*