



Lesson 29: Applying Tangents

Student Outcomes

- Students understand that the value of the tangent ratio of the angle of elevation or depression of a line is equal to the slope of the line.
- Students use the value of the tangent ratio of the angle of elevation or depression to solve real-world problems.

Lesson Notes

Lesson 26 introduced students to the tangent of θ as the ratio of the length of the opposite side to the length of the adjacent side. In Lesson 29, students use $\tan \theta$ in two different contexts: (1) the value of the ratio as it has to do with slope and (2) its use in solving real-world problems involving angles of elevation and depression. The focus of this lesson is modeling with mathematics, MP.4, as students work on a series of real-world applications with tangent.

Classwork

Opening Exercise (7 minutes)

For these exercises, consider asking half the class to work on part (a) and the other half to go directly to part (b), and then share results. Alternately, have students work in small groups. Students can work in pairs (or fours to split the work a little more); a student divides the values while the partner finds the tangent values, and they compare their results. Then, the class can debrief as a whole.

Opening Exercise

a. Use a calculator to find the tangent of θ . Enter the values, correct to four decimal places, in the last row of the table.

θ	0	10	20	30	40	50	60	70	80	90
$\sin \theta$	0	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848	1
$\cos \theta$	1	0.9848	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1736	0
$\frac{\sin \theta}{\cos \theta}$	0	0.1763	0.3639	0.5774	0.8392	1.1917	1.7320	2.7477	5.6728	undefined
$\tan \theta$	0	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713	undefined

Note to the teacher: Dividing the values in the first two rows provides a different answer than using the full decimal readout of the calculator.

MP.8

MP.8

- b. The table from Lesson 29 is provided here for you. In the row labeled $\frac{\sin \theta}{\cos \theta}$, divide the sine values by the cosine values. What do you notice?

For each of the listed degree values, dividing the value of $\sin \theta$ by the corresponding value of $\cos \theta$ yields a value very close to the value of $\tan \theta$. The values divided to obtain $\frac{\sin \theta}{\cos \theta}$ were approximations, so the actual values might be exactly the same.

Note to the teacher: Consider showing the calculation of $\frac{\sin \theta}{\cos \theta}$ using the calculator to reveal that the quotients are not just close to the value of $\tan \theta$, but are, in fact, equal to the value of $\tan \theta$.

Depending on student readiness, consider asking students to speculate what the relationship between sine, cosine, and tangent is. The relationship is summarized as $\frac{\frac{\text{opp}}{\text{adj}}}{\frac{\text{adj}}{\text{hyp}}} = \frac{\text{opp}}{\text{hyp}} \times \frac{\text{hyp}}{\text{adj}} = \frac{\text{opp}}{\text{adj}}$ but is expanded upon in Lesson 30. This level of explanation makes it clear why $\tan 0 = 0$ and why $\tan 90$ is undefined.

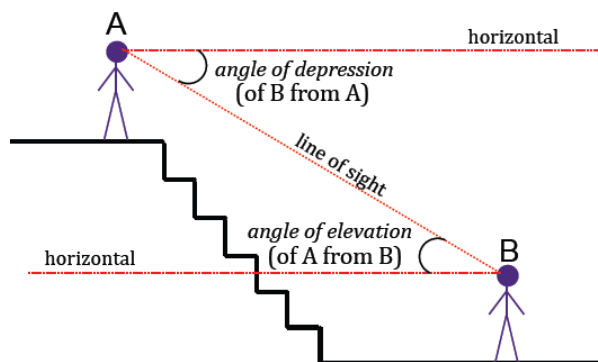
Discussion (5 minutes)

Lead students through a discussion that ties together the concepts of angle of elevation/depression in a real-world sense, and then, using the coordinate plane, tying it to tangent and slope.

- Consider the image below. How would you describe the angle of elevation? How would you describe the angle of depression?

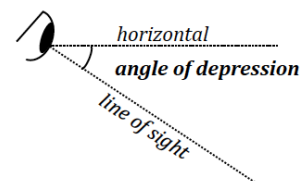
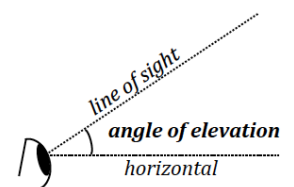
Allow students a moment to come up with their own definitions and then share out responses before presenting the following definition:

- The angle of elevation or depression is the angle between the horizontal (parallel with the earth’s surface) and the line of sight.
- In a case where two viewers can observe each other, such as in the diagram below, what do you notice about the measure of the angle of depression and the angle of elevation? Why?
 - They have the same measure since the horizontal from each viewer forms a pair of parallel lines, and the line of sight acts as a transversal; the angle of depression and angle of elevation are alternate interior angles of equal measure.



Scaffolding:

- Consider placing a poster of the following in a prominent place in the classroom:



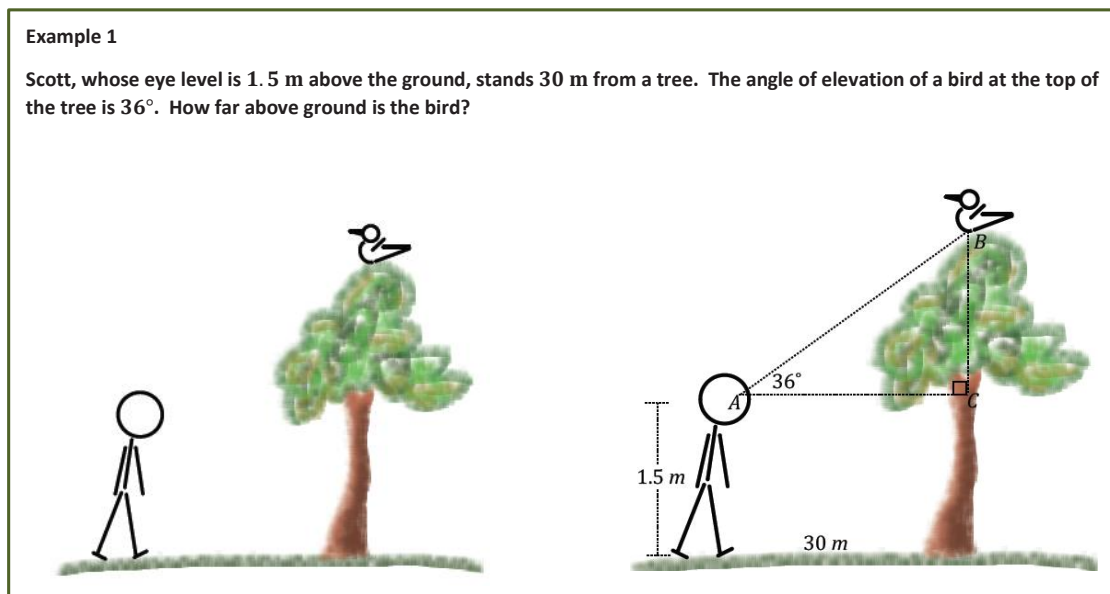
- Students may benefit from choral recitation of the description of each term.

Example 1 (7 minutes)

Example 1 is a real-world application of the angle of elevation and tangent. Encourage students to label the diagram as they read through the problem; consider using the same labels for the key vertices as a whole class. If students struggle to begin the problem, consider drawing the triangle for the class and then asking students to write or speak about how the triangle models the situation.

- Consider standing near a tree and observing a bird at the top of the tree. We can use measurements we know and the angle of elevation to help determine how high off the ground the bird is.

Please encourage students to attempt to solve first on their own or with a partner.



MP.1

- With respect to your diagram, think of the measurement you are looking for.
 - In our diagram, we are looking for BC .
- How will you find BC ?
 - I can use the tangent to determine BC in meters.

$$\tan 36 = \frac{BC}{AC}$$

$$\tan 36 = \frac{BC}{30}$$

$$30 \tan 36 = BC$$

$$BC \approx 21.8$$

- Have we found the height at which the bird is off the ground?
 - No. The full height must be measured from the ground, so the distance from the ground to Scott's eye level must be added to BC .
 - The height of the bird off of the ground is $1.5 \text{ m} + 21.8 \text{ m} = 23.3 \text{ m}$.

MP.1

- So, between the provided measurements, including the angle of elevation, and the use of the tangent ratio, we were able to find the height of the bird.

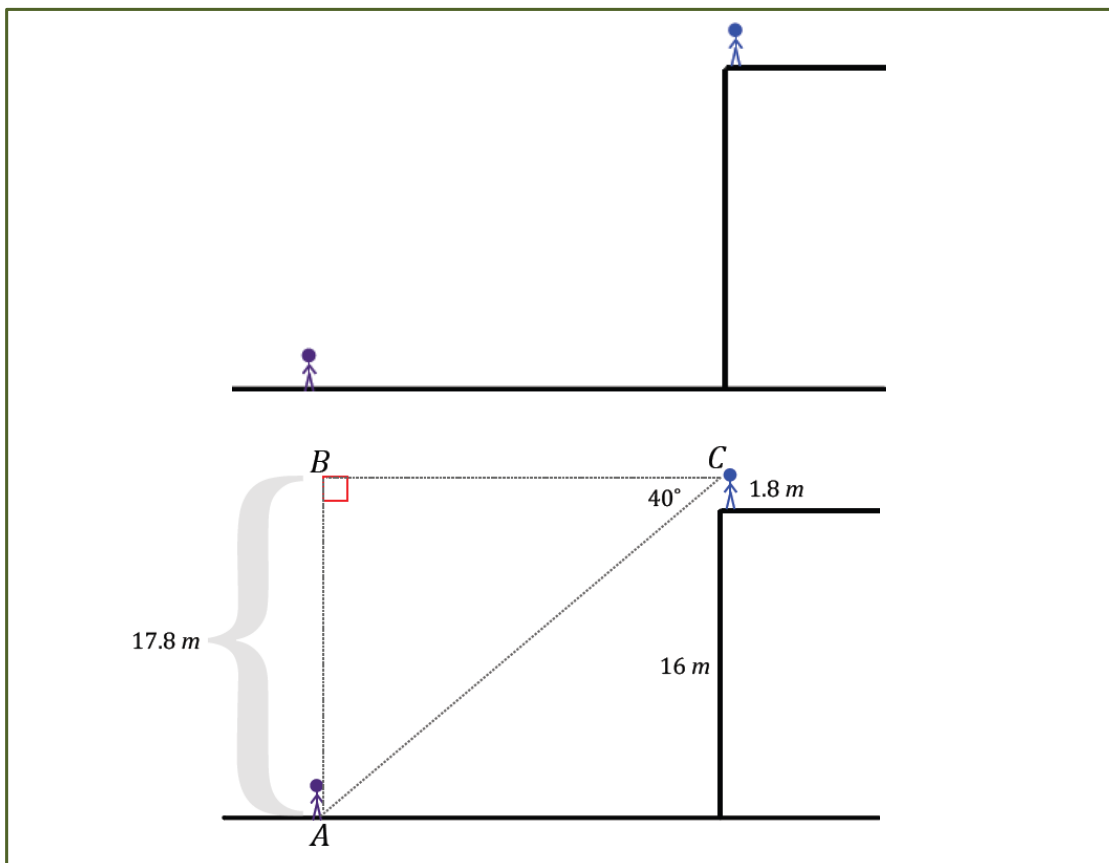
Example 2 (5 minutes)

Example 2 is a real-world application of the angle of depression and tangent. Encourage students to label the diagram as they read through the problem; consider having the whole class use common labels for the key vertices. Consider having students work with a partner for this example.

Example 2

From an angle of depression of 40° , John watches his friend approach his building while standing on the rooftop. The rooftop is 16 m from the ground, and John's eye level is at about 1.8 m from the rooftop. What is the distance between John's friend and the building?

Make sure to point out the angle of depression in the diagram below. Emphasize that the 40° angle of depression is the angle between the line of sight and the line horizontal (to the ground) from the eye.



- Use the diagram to describe the distance we must determine.
 - We are going to find BC in meters.

- How will we find BC ?

- We can use the tangent: $\tan 40 = \frac{AB}{BC}$.

$$\tan 40 = \frac{AB}{BC}$$

$$\tan 40 = \frac{17.8}{BC}$$

$$BC = \frac{17.8}{\tan 40}$$

$$BC \approx 21.2$$

- Again, with the assistance of a few measurements, including the angle of depression, we were able to determine the distance between John’s friend and the building, which is 21.2 meters.

Exercise 1 (4 minutes)

In the following problem, the height of the person is not provided as a piece of information. This is because the relative height of the person does not matter with respect to the other heights mentioned.

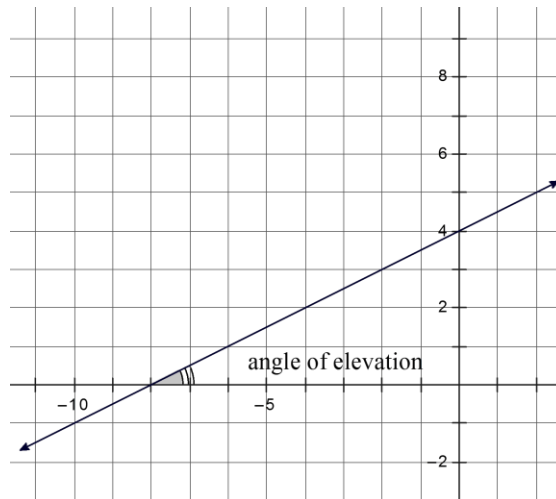
Exercise 1

Standing on the gallery of a lighthouse (the deck at the top of a lighthouse), a person spots a ship at an angle of depression of 20° . The lighthouse is 28 m tall and sits on a cliff 45 m tall as measured from sea level. What is the horizontal distance between the lighthouse and the ship? Sketch a diagram to support your answer.

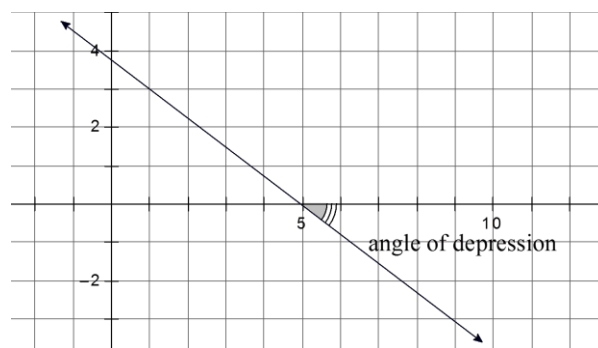
Approximately 201 m

Discussion (6 minutes)

- Now, let’s examine the angle of elevation and depression on the coordinate plane.
- Observe the following line, which has a positive slope and crosses the x -axis.
- The indicated angle is the angle of elevation.



- Notice that the slope of the line is the tangent of the angle of elevation. Why is this true?
 - *By building a right triangle (so that one leg is parallel to the x -axis and the other leg is parallel to the y -axis) around any two points selected on the line, we can determine the slope by finding the value of $\frac{\text{rise}}{\text{run}}$. This is equal to the tangent of the angle of elevation because, with respect to the angle of elevation, the tangent ratio is $\frac{\text{opp}}{\text{adj}}$, which is equivalent to the value of $\frac{\text{rise}}{\text{run}}$.*
- Similar to the angle of elevation, there is also an angle of depression. The indicated angle between the line declining to the right and the x -axis is the angle of depression.



- Notice that the tangent of the angle of depression will be positive; to accurately capture the slope of the line, we must take the negative of the tangent.

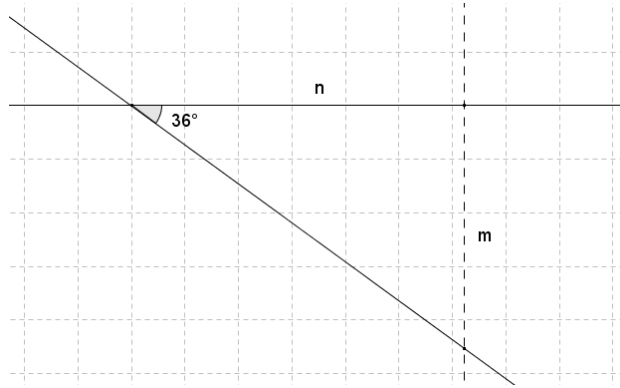
Exercise 2 (4 minutes)

Exercise 2

A line on the coordinate plane makes an angle of depression of 36° . Find the slope of the line correct to four decimal places.

Choose a segment on the line, and construct legs of a right triangle parallel to the x - and y -axes, as shown. If m is the length of the vertical leg and n is the length of the horizontal leg, then $\tan 36 = \frac{m}{n}$. The line decreases to the right, so the value of the slope must be negative. Therefore,

$$\text{slope} = -\frac{m}{n} = -\tan 36 \approx -0.7265.$$



Closing (2 minutes)

Ask students to summarize the key points of the lesson. Additionally, consider asking students the following questions independently in writing, to a partner, or to the whole class.

- Review the relationship between tangent and slope.
 - The slope of a line increasing to the right is the same as the tangent of the angle of elevation of the line. The slope of a line decreasing to the right is the same as the negative tangent of the angle of depression of the line.
- Review the relationship between tangent and slope.
- Review what the angle of elevation and depression means in a real-world context.
 - The angle of elevation or depression is the angle between the horizontal and the line of sight.

Exit Ticket (5 minutes)

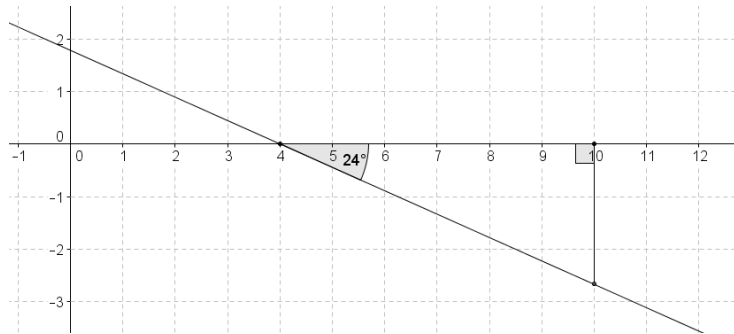
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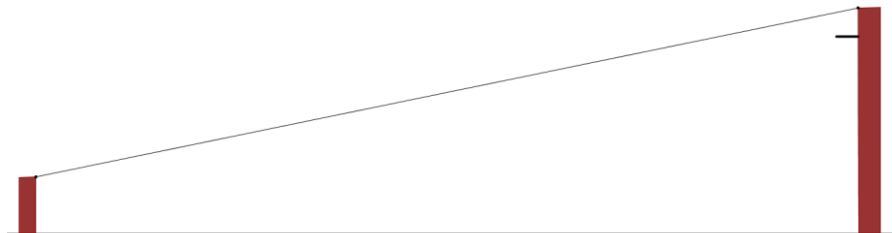
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Exit Ticket

- The line on the coordinate plane makes an angle of depression of 24° . Find the slope of the line correct to four decimal places.



- Samuel is at the top of a tower and will ride a trolley down a zip line to a lower tower. The total vertical drop of the zip line is 40 ft. The zip line's angle of elevation from the lower tower is 11.5° . What is the horizontal distance between the towers?

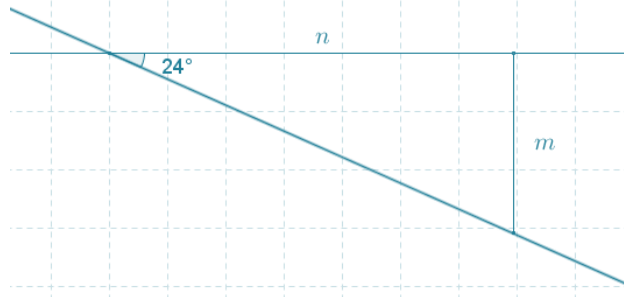


Exit Ticket Sample Solutions

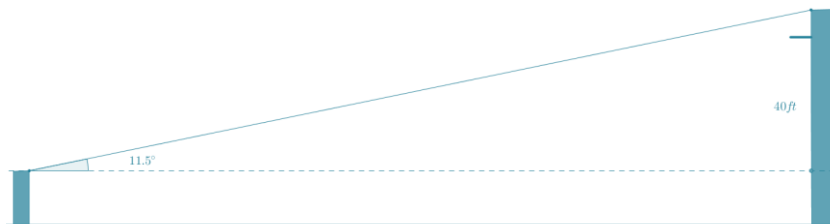
1. A line on the coordinate plane makes an angle of depression of 24° . Find the slope of the line correct to four decimal places.

Choose a segment on the line, and construct legs of a right triangle parallel to the x - and y -axes, as shown. If m is the length of the vertical leg and n is the length of the horizontal leg, then $\tan 24 = \frac{m}{n}$. The line decreases to the right, so the value of the slope must be negative. Therefore,

$$\text{slope} = -\frac{m}{n} = -\tan 24 \approx -0.4452.$$



2. Samuel is at the top of a tower and will ride a trolley down a zip line to a lower tower. The total vertical drop of the zip line is 40 ft. The zip line's angle of elevation from the lower tower is 11.5° . What is the horizontal distance between the towers?



A right triangle is formed by the zip line's path, the vertical drop along the upper tower, and the horizontal distance between the towers. Let x represent the horizontal distance between the towers in feet. Using tangent:

$$\begin{aligned} \tan 11.5 &= \frac{40}{x} \\ x &= \frac{40}{\tan 11.5} \\ x &\approx 196.6 \end{aligned}$$

The horizontal distance between the towers is approximately 196.6 ft.

Problem Set Sample Solutions

1. A line in the coordinate plane has an angle of elevation of 53° . Find the slope of the line correct to four decimal places.

Since parallel lines have the same slope, we can consider the line that passes through the origin with an angle of inclination of 53° . Draw a vertex at $(5, 0)$ on the x -axis, and draw a segment from $(5, 0)$ parallel to the y -axis to the intersection with the line to form a right triangle.

If the base of the right triangle is 5-units long, let the height of the triangle be represented by y .

$$\tan 53 = \frac{x}{5}$$

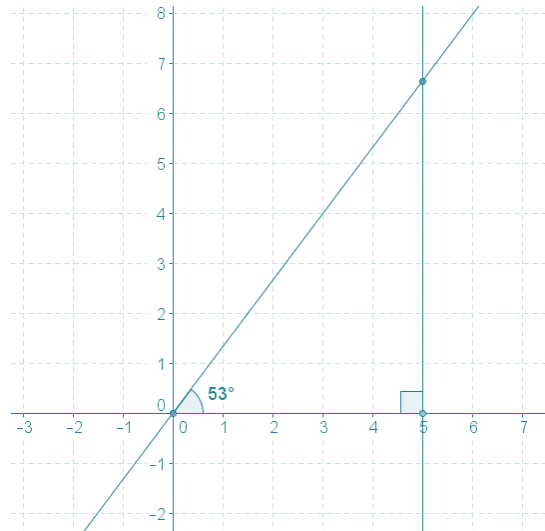
$$5(\tan 53) = x$$

The slope of the line is

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$\text{slope} = \frac{5(\tan 53)}{5}$$

$$\text{slope} = \tan 53 \approx 1.3270.$$



2. A line in the coordinate plane has an angle of depression of 25° . Find the slope of the line correct to four decimal places.

A line crosses the x -axis, decreasing to the right with an angle of depression of 25° . Using a similar method as in Problem 1, a right triangle is formed. If the leg along the x -axis represents the base of the triangle, let h represent the height of the triangle.

Using tangent:

$$\tan 25 = \frac{h}{5}$$

$$5(\tan 25) = h$$

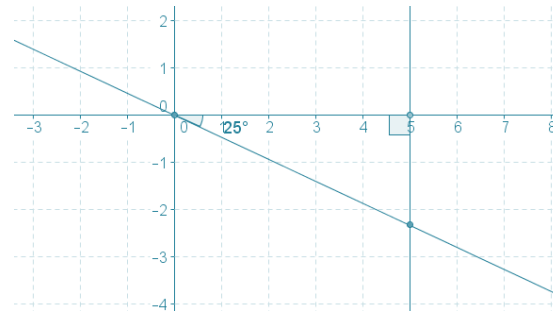
The slope of the line is negative since the line decreases to the right:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$\text{slope} = -\frac{h}{5}$$

$$\text{slope} = -\frac{5(\tan 25)}{5}$$

$$\text{slope} = -\tan 25 \approx -0.4663$$



3. In Problems 1 and 2, why do the lengths of the legs of the right triangles formed not affect the slope of the line?

When using the tangent ratio, the length of one leg of a right triangle can be determined in terms of the other leg. Let x represent the length of the horizontal leg of a slope triangle. The vertical leg of the triangle is then $x \tan \theta$, where θ is the measure of the angle of inclination or depression. The slope of the line is $\frac{\text{rise}}{\text{run}} = \frac{x \tan \theta}{x} = \tan \theta$.

4. Given the angles of depression below, determine the slope of the line with the indicated angle correct to four decimal places.

a. 35° angle of depression

$$\tan 35 \approx 0.7002$$

$$\text{slope} \approx -0.7002$$

b. 49° angle of depression

$$\tan 49 \approx 1.1504$$

$$\text{slope} \approx -1.1504$$

c. 80° angle of depression

$$\tan 80 \approx 5.6713$$

$$\text{slope} \approx -5.6713$$

d. 87° angle of depression

$$\tan 87 \approx 19.0811$$

$$\text{slope} \approx -19.0811$$

e. 89° angle of depression

$$\tan 89 \approx 57.2900$$

$$\text{slope} \approx -57.2900$$

f. 89.9° angle of depression

$$\tan 89.9 \approx 572.9572$$

$$\text{slope} \approx -572.9572$$

g. What appears to be happening to the slopes (and tangent values) as the angles of depression get closer to 90°?

As the angles get closer to 90°, their slopes (and tangent values) get much farther from zero.

h. Find the slopes of angles of depression that are even closer to 90° than 89.9°. Can the value of the tangent of 90° be defined? Why or why not?

Choices of angles will vary. The closer an angle is in measure to 90°, the greater the tangent value of that angle and the farther the slope of the line determined by that angle is from zero. An angle of depression of 90° would be a vertical line, and vertical lines have 0 run; therefore, the value of the ratio rise:run is undefined. The value of the tangent of 90° would have a similar outcome because the adjacent leg of the "triangle" would have a length of 0, so the ratio $\frac{\text{opp}}{\text{adj}} = \frac{\text{opp}}{0}$, which is undefined.

5. For the indicated angle, express the quotient in terms of sine, cosine, or tangent. Then, write the quotient in simplest terms.

a. $\frac{4}{2\sqrt{13}}; \alpha$

$$\cos \alpha = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

b. $\frac{6}{4}; \alpha$

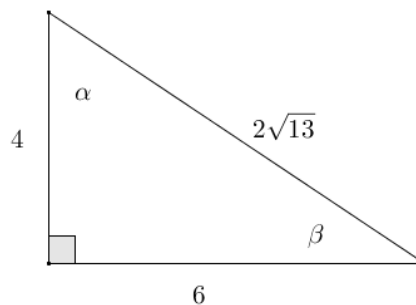
$$\tan \alpha = \frac{6}{4} = \frac{3}{2}$$

c. $\frac{4}{2\sqrt{13}}; \beta$

$$\sin \beta = \frac{4}{2\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

d. $\frac{4}{6}; \beta$

$$\tan \beta = \frac{4}{6} = \frac{2}{3}$$



6. The pitch of a roof on a home is expressed as a ratio of vertical rise: horizontal run where the run has a length of 12 units. If a given roof design includes an angle of elevation of 22.5° and the roof spans 36 ft. as shown in the diagram, determine the pitch of the roof. Then, determine the distance along one of the two sloped surfaces of the roof.

The diagram as shown is an isosceles triangle since the base angles have equal measure. The altitude, a , of the triangle is the vertical rise of the roof.

The right triangles formed by drawing the altitude of the given isosceles triangle have a leg of length 18 ft.

$$\tan 22.5 = \frac{a}{18}$$

$$a = 18 \tan 22.5$$

$$a \approx 7.5$$

Roof pitch:

$$\frac{7.5}{18} = \frac{h}{12}$$

$$18h = 12 \cdot 7.5$$

$$h = 5$$

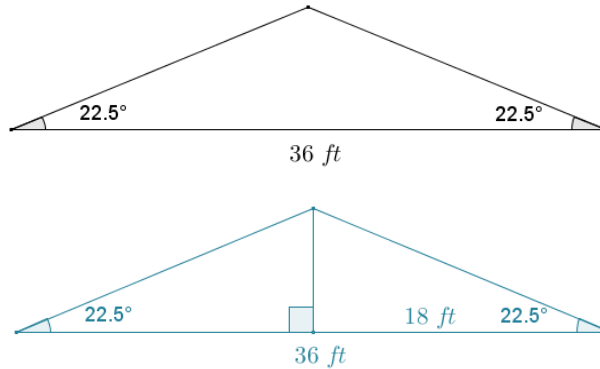
$$\cos 22.5 = \frac{18}{s}$$

$$s = \frac{18}{\cos 22.5}$$

$$s \approx 19.5$$

The pitch of the roof is 5:12.

The sloped surface of the roof has a distance of approximately 19.5 ft.



7. An anchor cable supports a vertical utility pole forming a 51° angle with the ground. The cable is attached to the top of the pole. If the distance from the base of the pole to the base of the cable is 5 meters, how tall is the pole?

Let h represent the height of the pole in meters.

Using tangent:

$$\tan 51 = \frac{h}{5}$$

$$5(\tan 51) = h$$

$$6.17 \approx h$$

The height of the utility pole is approximately 6.17 meters.

8. A winch is a tool that rotates a cylinder, around which a cable is wound. When the winch rotates in one direction, it draws the cable in. Joey is using a winch and a pulley (as shown in the diagram) to raise a heavy box off the floor and onto a cart. The box is 2 ft. tall, and the winch is 14 ft. horizontally from where cable drops down vertically from the pulley. The angle of elevation to the pulley is 42° . What is the approximate length of cable required to connect the winch and the box?

Let h represent the length of cable in the distance from the winch to the pulley along the hypotenuse of the right triangle shown in feet, and let y represent the distance from the pulley to the floor in feet.

Using tangent:

$$\tan 42 = \frac{y}{14}$$

$$14(\tan 42) = y$$

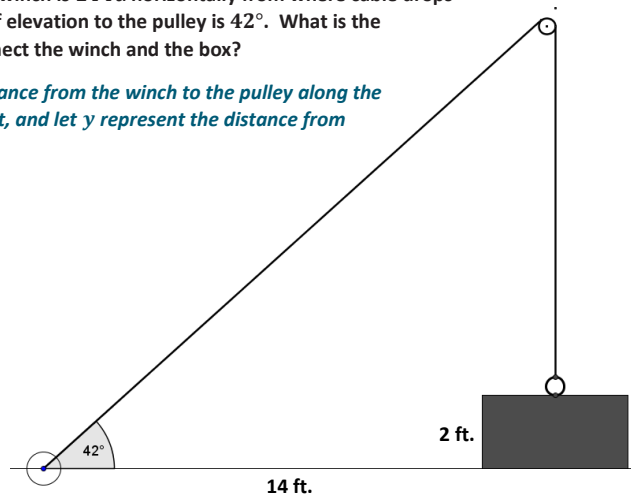
$$12.61 \approx y$$

Using cosine:

$$\cos 42 = \frac{14}{h}$$

$$h = \frac{14}{\cos 42}$$

$$h \approx 18.84$$



Let t represent the total amount of cable from the winch to the box in feet:

$$t \approx 12.61 + 18.84 - 2$$

$$t \approx 29.45$$

The total length of cable from the winch to the box is approximately 29.45 ft.