Name ____ Date ____________________

1.

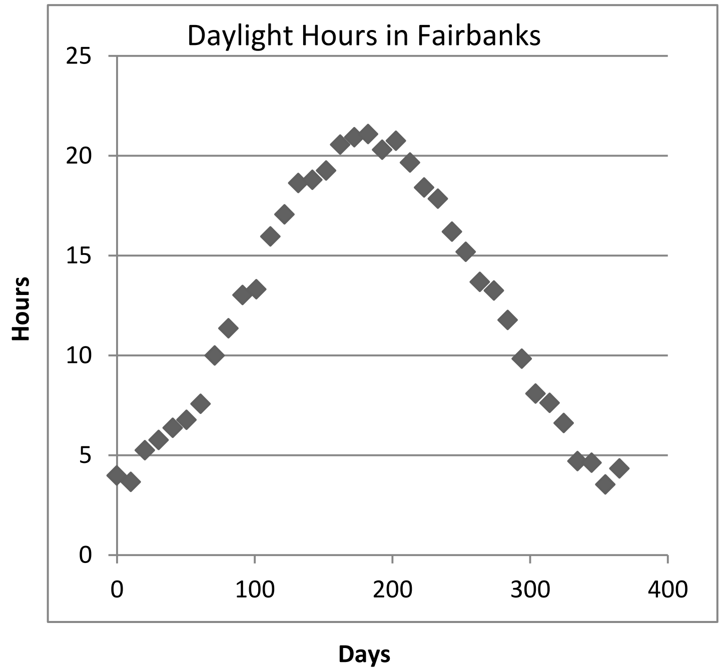
a. Graph the function $f(x) = 3\cos(2x) + 1$ between 0 and 2π.

b. Graph and label the midline on your graph. Draw and label a segment to represent the period and specify its length.

c. Explain how you can find the midline, period, and amplitude in part (b) from the function $f(x) = 3\cos(2x) + 1$.

d. Construct a periodic function that has period 8π, a midline given by the equation $y = 5$, and an amplitude of $\frac{1}{2}$.

2. The graph below shows the number of daylight hours each day of the year in Fairbanks, Alaska, as a function of the day number of the year. (January 1 is day 1, January 2 is day 2, and so on.)



a. Find a function that models the shape of this daylight-hour curve reasonably well. Define the variables you use.

b. Explain how you chose the numbers in your function from part (a): What is the midline? What is the amplitude? What is the period?

c. A friend looked at the graph and wondered, "What was the average number of daylight hours in Fairbanks over the past year?" What might be a reasonable answer to that question? Use the structure of the function you created in part (a) to explain your answer.

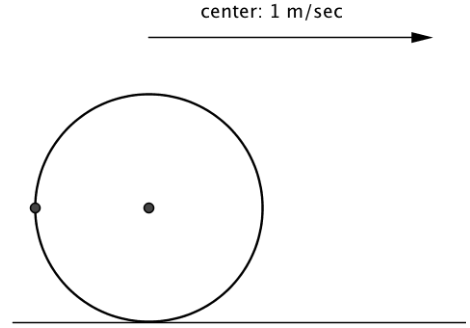
d. According to the graph, around which month of the year did the first day of the year with $17\frac{1}{2}$ hours of daylight occur? Does your function in part (a) agree with your estimation?

e. The scientists who reported these data now inform us that their instruments were incorrectly calibrated; each measurement of the daylight hours is 15 minutes too long. Adjust your function from part (a) to account for this change in the data. How does your function now appear? Explain why you changed the formula as you did.

f. To make very long-term predictions, researchers would like a function that acknowledges that there are, on average, $365\frac{1}{4}$ days in a year. How should you adjust your function from part (e) so that it represents a function that models daylight hours with a period of $365\frac{1}{4}$ days? How does your function now appear?

g. Do these two adjustments to the function significantly change the prediction as to which day of the year first possesses $17\frac{1}{2}$ hours of daylight?

3. On a whim, James challenged his friend Susan to model the movement of a chewed-up piece of gum stuck to the rim of a rolling wheel with radius 1 m. To simplify the situation, Susan drew a diagram of a circle to represent the wheel and imagined the gum as a point on the circle. Furthermore, she assumed that the center of the wheel was moving to the right at a constant speed of 1 m/sec, as shown in the diagram.



At time $t = 0$ seconds, the piece of gum was directly to the left of the center of the wheel, as indicated in the diagram above.

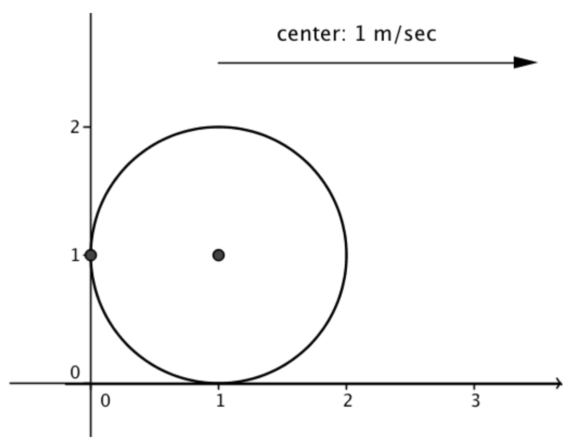
a. What is the first time that the gum was at the top position of the wheel?

b. What is the first time that the gum was again directly to the left of the center of the wheel?

c. After doing some initial calculations as in parts (a) and (b), Susan realized that the height of the gum is a function of time. She let $V(t)$ stand for the vertical height of the gum from the ground at time t seconds. Find a formula for her function.

d. What is the smallest positive value of t for which $V(t) = 0$? What does this value of t represent in terms of the situation?

Next, Susan imagined that the wheel was rolling along the horizontal axis of a coordinate system, with distances along the horizontal axis given in units of meters (and height along the vertical axis also given in units of meter). At time $t = 0$, the center of the wheel has coordinates $(1, 1)$ so that the gum was initially at position $(0, 1)$.



e. What is the x-coordinate of the position of the gum after $\frac{\pi}{2}$ seconds (when it first arrived at the top of the wheel)? After π seconds (when it was directly to the right of the center)?

f. From the calculations like those in part (e), Susan realized that the horizontal distance, H, of the gum from its initial location is also a function of time t, given by the distance the wheel traveled plus its horizontal displacement from the center of the wheel. Write a formula $H(t)$ for the function (i.e., find a function that specifies the x-coordinate of the position of the gum at time t).

g. Susan and James decide to test Susan's model by actually rolling a wheel with radius 1 m. However, when the gum first touched the ground, it came off the wheel and stuck to the ground at that position. How horizontally far from the initial position is the gum? Verify that your function from part (f) predicts this answer, too.

4. Betty was looking at the Pythagorean Identity: for all real numbers θ,

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

a. Betty used the Pythagorean identity to make up the equation below. She then stated, "Wow, I've discovered a new identity that is true for all θ." Do you agree with her? Why or why not?

$$\frac{\sin^2(\theta)}{1 - \cos(\theta)} = 1 + \cos(\theta)$$

b. Prove the Pythagorean identity.

c. The real number θ is such that $\sin(\theta) = 0.6$. Calculate $|\cos(\theta)|$ and $|\tan(\theta)|$.

d. Suppose additional information is given about the number θ from part (c). You are told that $\frac{\pi}{2} < \theta < \pi$. What are the values of $\cos(\theta)$ and $\tan(\theta)$? Explain.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a** **F-IF.C.7e**	Student does not draw the graph of a periodic function.	Student draws the graph of one function correctly.	Student draws the graphs of two functions correctly.	Student draws the graphs of all three functions correctly.
	b **F-IF.C.7e**	Student indicates neither midline nor period.	Student correctly indicates midline and/or period, but neither is labeled by the length of the period or the equation of the midline.	Student correctly draws midline and period segment, but only one is labeled correctly by the length of the period or equation of the midline.	Student correctly draws midline and period segment, and they are labeled by the correct length of the period/equation of the midline.
	c **F-IF.C.7e**	Student does not explain how to obtain any of the quantities.	Student correctly finds at least one of the quantities.	Student correctly finds all but one of the quantities.	Student correctly finds values for all the quantities and understands the relationship between the constant ω and the period.

	d **F-IF.C.7e**	Student only guesses that a sine or cosine function is needed.	Student knows that a sine or cosine function will work but does not obtain the correct parameters.	Student correctly sketches the graph but misses one of the parameters or has the graph incorrectly shifted. OR Student sketches the graph adequately but not the correct relationship between the constant ω and the period.	Student provides the correct values for each of the parameters, understands that either a sine or cosine will work, and presents the correct relationship between the constant ω and the period.
2	**a** **S-ID.6a**	Student does not use a trigonometric function or uses one other than sine or cosine.	Student understands that the answer is related to the sine or cosine function but does not estimate the midline, amplitude, or period correctly and does not provide the correct function.	Student estimates the midline, amplitude, and period of the trigonometric function adequately but does not provide the correct function.	Student provides the correct function, understands the form of the answer, and correctly determines the amplitude, period, and constant. Student understands that there are different sine and cosine representations for the same function. $H(d) = -\frac{17}{2}\cos\left(\frac{2\pi}{365}d\right) + 12\frac{1}{2}$
	b **F-TF.B.5** **F-IF.7e**	Student does not describe any of the values corresponding to midline, amplitude, or period.	Student describes one of the three values corresponding to midline, amplitude, or period.	Student describes two of the three values corresponding to midline, amplitude, or period.	Student describes each of the three values corresponding to midline, amplitude, and period. Midline: $y = 12\frac{1}{2}$ Amplitude: $\frac{17}{2}$ Period: 365
	c **S-ID.6a** **F-IF.7e**	Student does not understand the concept of midline.	Student selects a point on the function or uses the value 0.	Student estimates the midline but does not explain that he is using the midline.	Student provides an adequate approximation of the midline within a reasonable margin of error and states that the midline was used to estimate the average.
	d **F-TF.B.5**	Student provides a poor estimate for the number of days.	Student provides a reasonable estimate but does not justify it.	Student provides a reasonable estimate but does not correctly substitute the estimate into the function.	Student provides a reasonable estimate and substitutes the estimate into the function to get a number of hours that is relatively close to 17.5.

	e **F-TF.B.5**	Student does not know where to include the information to modify the trigonometric function.	Student does not display understanding of which parameter needs to be changed.	Student adds instead of subtracts to modify the added constant.	Student correctly understands how the change in data affects the function. $H(d) = -\frac{17}{2}\cos\left(\frac{2\pi}{365}d\right) + 12\frac{1}{4}$
	f **F-TF.B.5**	Student does not know where to include the information to modify the trigonometric function.	Student does not display understanding of which parameter needs to be changed.	Student understands that the calculation relates to the period but does not correctly change the period.	Student correctly understands how to calculate the new period. $H(d) = -\frac{17}{2}\cos\left(\frac{2\pi}{365.25}d\right) + 12\frac{1}{4}$
	g **F-TF.B.5** **S-ID.6a**	Student uses a modified formula incorrectly.	Student does not use the correct modified formula. OR Student uses the correct modified formula but does not calculate the trigonometric function correctly.	Student uses the correct formula but substitutes the wrong value. OR Student correctly evaluates the new value but does not provide an adequate criterion for determining if the change is significant.	Student correctly evaluates the new value and has a reasonable criterion for determining if it is a significant change.
3	**a** **F-TF.B.5**	Student does not see the relationship between time and the position of the gum or arc length and the position of the gum.	Student uses degrees instead of radians.	Student uses radians but models the circle incorrectly or mixes up the units, such as providing the answer in radians instead of in units of time.	Student determines the correct time that the gum is at the top of the wheel where the center is fixed (i.e., $\frac{\pi}{2}$ seconds).
	b **F-TF.B.5**	Student does not correctly measure a complete revolution of the circle (wheel).	Student uses degrees instead of radians or uses the wrong number of rotations.	Student mixes up clockwise and counterclockwise. However, the student understands that the circle has done one revolution and correctly uses radian measures.	Student correctly extends the formula from part (a) to part (b) and determines that the gum is to the left of the wheel at 2π seconds.
	c **F-TF.B.5**	Student does not use a trigonometric function. OR Student appears to guess which function.	Student understands that the vertical height is related to the sine or cosine function.	Student understands that the vertical height should be modeled with a sine function but does not use the function correctly or does not account for the position of the center of the wheel.	Student correctly models the vertical height ($V(t)$) with a sine function and adds the correct constant to reflect the position of the center of the wheel. $V(t) = 1 + \sin(t)$

	d **F-TF.B.5**	Student does not know what to solve for or what formula to use.	Student sets a formula equal to zero, but it is the wrong formula. OR Student does not know how to obtain the solution.	Student correctly gets the right formula but has a solution that is off by a multiple of $\frac{\pi}{2}$. OR Student has the correct value with the wrong units.	Student correctly solves the equation and understands the units (i.e., $\frac{3\pi}{2}$ seconds).
	e **F-TF.B.5**	Student does not understand that trigonometric functions are needed.	Student does not understand where the center of the wheel is at the specified time.	Student correctly calculates the position of the center of the wheel at this time but has the wrong angle.	Student understands the relation of the center of the wheel to the position (of the gum) at the two particular values of t (i.e., $1 + \frac{\pi}{2}$ meters and $2 + \pi$ meters).
	f **F-TF.B.5**	Student does not understand that trigonometric functions are needed.	Student does not see that the x-coordinate and y-coordinate have to be calculated separately but knows the calculations are related to sines and cosines.	Student correctly calculates the x-coordinate of the gum on a nonmoving wheel but does not understand how to extend this to the moving wheel. Student understands where the center of the wheel is at time t.	Student understands the relation of the center of the wheel to the gum and correctly uses the formulas obtained from the wheel with a fixed center.
	g **F-TF.B.5**	Student does not correctly place the center of the wheel at the specified time.	Student has either the gum or the wheel in the correct place.	Student solves the problem but does not have the general formula for the position of the gum. OR Student understands that the problem can be solved two ways but gets different answers.	Student understands that the problem can be solved using just the rotation of the wheel without trigonometric functions and, with the trigonometric functions and the two methods, gives the same answer.
4	**a** **F-TF.C.8**	Student does not know what a trigonometric identity is.	Student tests the formula for different values of x and determines that it is an identity.	Student determines that it is a trigonometric identity and correctly does the algebra but neglects to consider that the denominator is sometimes zero.	Student proves that the two sides match when the cosine is not zero and clearly understands that it is not an identity because the domains do not match.

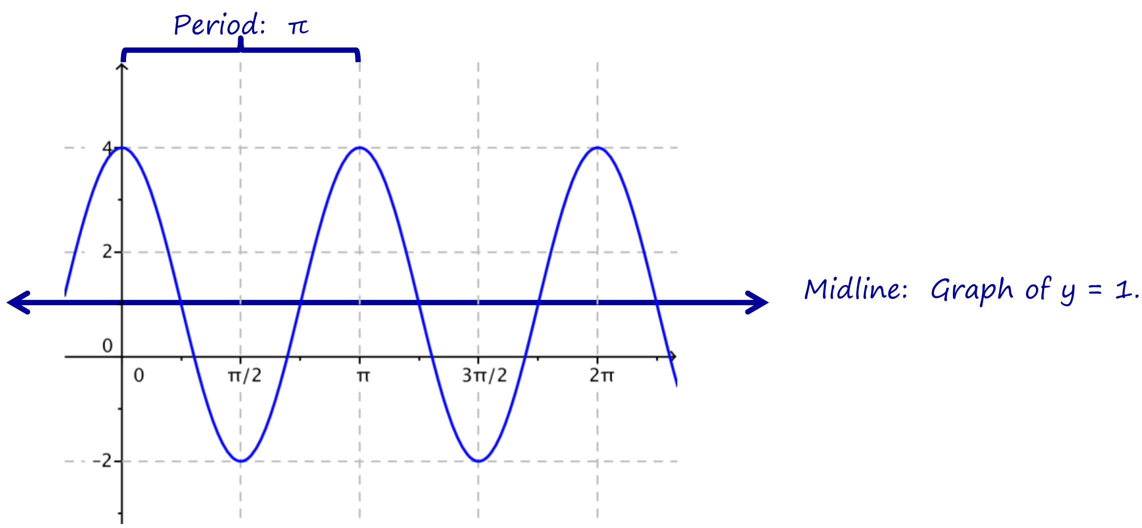
	b **F-TF.C.8**	Student does not know what a trigonometric identity is.	Student tests the formula for different values of x and determines that it is an identity.	Student provides an incorrect proof of the Pythagorean identity.	Student provides a correct proof.
	c **F-TF.C.8**	Student substitutes the value for $\sin(\theta)$ but cannot solve the equation.	Student substitutes the correct value for $\sin(\theta)$ and solves the equation but only gets one value for the cosine and does not know how to determine the tangent.	Student substitutes the correct value for $\sin(\theta)$ and solves the equation but only gets one value for the cosine. Student is able to determine the value of the tangent. OR Student correctly determines both values for the cosine and is unable to determine either value of the tangent.	Student correctly determines the values for the cosine and the tangent and understands that there are two solutions (i.e., $\lvert\cos(\theta)\rvert = 0.8$, $\lvert\tan(\theta)\rvert = 0.75$).
	d **F-TF.C.8**	Student does not understand that only the signs need to be changed from problem (c).	Student uses the correct signs but the wrong numerical value. OR Student calculates reciprocals.	Student provides the correct (absolute) value, but one sign is incorrect.	Student provides the correct sign and value (i.e., $\cos(\theta) = -0.8$, $\tan(\theta) = -0.75$).

Name ______________________________ Date ______________

1.

a. Graph the function $f(x) = 3\cos(2x) + 1$ between 0 and 2π.

Note that the figure below includes the response to part (b).



b. Graph and label the midline on your graph. Draw and label a segment to represent the period and specify its length.

c. Explain how you can find the midline, period, and amplitude in part (b) from the function $f(x) = 3\cos(2x) + 1$.

The midline is $y = 1$, where 1 is the constant added to the cosine function; the period satisfies the equation

$$2 = \frac{2\pi}{p}$$

d. Construct a periodic function that has period 8π, a midline given by the equation $y = 5$, and an amplitude of $\frac{1}{2}$.

A sine or cosine function will work. Using the sine for our solution, we note that for the function in the following form:

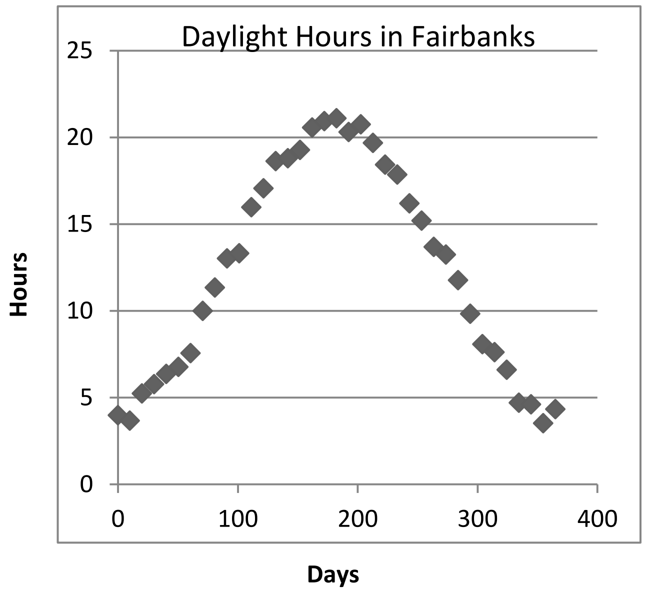
$$f(x) = A\sin(\omega x) + c.$$

A is the amplitude, c is the vertical shift, and the period p satisfies $\frac{2\pi}{p} = \omega$. *So,*

$$f(x) = \frac{1}{2}\sin\left(\frac{1}{4}x\right) + 5$$

will work. Replacing the sine with cosine works equally well.

2. The graph below shows the number of daylight hours each day of the year in Fairbanks, Alaska, as a function of the day number of the year. (January 1 is day 1, January 2 is day 2, and so on.)



a. Find a function that models the shape of this daylight-hour curve reasonably well. Define the variables you use.

Once we have the midline and realize that this is a periodic function, we expect the curve to be a sine or cosine function. It looks like an upside-down cosine graph, so the function will be in the form $H(d) = -A\cos(\omega d) + k$, where d is the number of days, k is the height of the midline, A is the distance between the peak (which looks to be about 21) and the midline, and $P = \frac{2\pi}{\omega}$ is the period.

So, we have

$$A = 21 - 12\frac{1}{2} = \frac{17}{2};\ \omega = \frac{2\pi}{365};\ k = 12\frac{1}{2}.$$

$$H(d) = -\frac{17}{2}\cos\left(\frac{2\pi}{365}d\right) + 12\frac{1}{2}.$$

Here, the function H is the number of hours of daylight (in units of hours), and d is the day number as defined at the beginning of the question.

[NOTE: Variations of this formula such as a sine function with a phase shift are possible.]

b. Explain how you chose the numbers in your function from part (a): What is the midline? What is the amplitude? What is the period?

The midline is the horizontal line that is halfway between a maximum and minimum value, so it corresponds to the graph of $y = 12\frac{1}{2}$. The amplitude is described by the distance from a maximum value and the midline, which corresponds to $\frac{17}{2}$. The period is the horizontal distance between two sequential minimums, which in this problem corresponds to 365.

c. A friend looked at the graph and wondered, "What was the average number of daylight hours in Fairbanks over the past year?" What might be a reasonable answer to that question? Use the structure of the function you created in part (a) to explain your answer.

The average number of daylight hours was $12\frac{1}{2}$. The average number of daylight hours appears to be given by the midline of the graph of the function, which is given by the value of k in the answer to part (a), that is $12\frac{1}{2}$.

d. According to the graph, around which month of the year did the first day of the year with 17.5 hours of daylight occur? Does your function in part (a) agree with your estimation?

According to the graph, 17.5 hours of daylight first occurred around day 130. We can check the estimation by substituting it into the function:

$$H(130) = -\frac{17}{2}\cos\left(\frac{2\pi}{365}\cdot 130\right) + 12\frac{1}{2} = 17.8.$$

The value of H(130) is 17.8 hours of light, which is pretty close to the number of daylight hours estimated from the graph.

Note: An exact answer is not needed here. Likely reasonable answers include: 120, 125, 130, 133, or anything in between.

e. The scientists who reported these data now inform us that their instruments were incorrectly calibrated; each measurement of the daylight hours is 15 minutes too long. Adjust your function from part (a) to account for this change in the data. How does your function now appear? Explain why you changed the formula as you did.

To correct the measurement, you need to take away $\frac{1}{4}$ hours of daylight from each measurement so that it lowers the midline by $\frac{1}{4}$ hours: $H(d) = -\frac{17}{2}\cos\left(\frac{2\pi}{365}d\right) + 12\frac{1}{4}$.

f. To make very long-term predictions, researchers would like a function that acknowledges that there are, on average, $365\frac{1}{4}$ days in a year. How should you adjust your function from part (e) so that it represents a function that models daylight hours with a period of $365\frac{1}{4}$ days? How does your function now appear?

This lengthens the period by $\frac{1}{4}$ day; $H(d) = -\frac{17}{2}\cos\left(\frac{2\pi}{365.25}d\right) + 12\frac{1}{4}$.

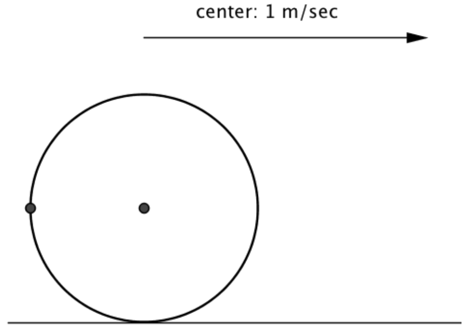
g. Do these two adjustments to the function significantly change the prediction as to which day of the year first possesses 17.5 hours of daylight?

Using the new function and repeating the calculations as in part (d), we have

$$H(130) = -\frac{17}{2}\cos\left(\frac{2\pi}{365.25}130\right) + 12\frac{1}{4} \approx 17.5,$$

which is not a great change between the two predictions (the difference is less than 0.3 hours), but the function definitely seems to be more accurate with this additional information.

3. On a whim, James challenged his friend Susan to model the movement of a chewed-up piece of gum stuck to the rim of a rolling wheel with radius 1 m. To simplify the situation, Susan drew a diagram of a circle to represent the wheel and imagined the gum as a point on the circle. Furthermore, she assumed that the center of the wheel was moving to the right at a constant speed of 1 m/sec, as shown in the diagram.



At time $t = 0$ seconds, the piece of gum was directly left of the center of the wheel, as indicated in the diagram above.

a. What is the first time that the gum was at the top position of the wheel?

Since the wheel was rolling at 1 meter per second, it made a complete turn after 2π seconds, rotating clockwise at a constant rate of 1 radian per second. Hence, the top position was reached after the gum has rotated $\frac{\pi}{2}$ radians, which occurs at time $\frac{\pi}{2}$ seconds.

b. What is the first time that the gum was again directly to the left of the center of the wheel?

The gum did one complete turn, which occurred after the gum had rotated 2π radians, which is at time 2π seconds.

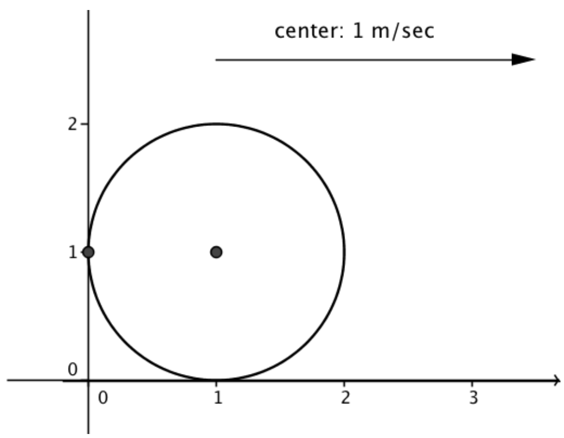
c. After doing some initial calculations as in parts (a) and (b), Susan realized that the height of the gum is a function of time. She let $V(t)$ stand for the vertical height of the gum from the ground at time t seconds. Find a formula for her function.

$V(t) = 1 + \sin(\pi - t)$ or $V(t) = 1 + \sin(t)$.

d. What is the smallest positive value of t for which $V(t) = 0$? What does this value of t represent in terms of the situation?

We want the first positive number that satisfies $1 + \sin(t) = 0$. This occurs when $t = \frac{3\pi}{2}$. At $\frac{3\pi}{2}$ seconds, the gum will be on the ground.

Next, Susan imagined that the wheel was rolling along the horizontal axis of a coordinate system, with distances along the horizontal axis given in units of meters (and height along the vertical axis also given in units of meter). At time $t = 0$, the center of the wheel has coordinates $(1, 1)$ so that the gum was initially at position $(0, 1)$.



e. What is the x-coordinate of the position of the gum after $\frac{\pi}{2}$ seconds (when it first arrived at the top of the wheel)? After π seconds (when it was directly to the right of the center)?

The wheel has moved forward $\frac{\pi}{2}$ meters, and the point, with the respect to the wheel, has moved one unit to the right. Its x-coordinate is thus $1 + \frac{\pi}{2}$ meters. Similarly, after π seconds, the x-coordinate of the gum was $(1 + \pi) + 1$ meters away from the starting position, which was $2 + \pi$ meters.

f. From the calculations like those in part (e), Susan realized that the horizontal distance, H, of the gum from its initial location is also a function of time t, given by the distance the wheel traveled plus its horizontal displacement from the center of the wheel. Write a formula $H(t)$ for the function (i.e., find a function that specifies the x-coordinate of the position of the gum at time t).

After t seconds, the center of the wheel is at position $1 + t$ meters. We add this distance to the horizontal displacement that the gum is away from the center (the gum is horizontally moving back and forth with respect to the center). The horizontal displacement can be modeled by $-\cos(t)$ (or equivalently, $\cos(\pi - t)$). Adding the two terms together specifies the x-coordinate of the gum at time t:

$H(t) = 1 + t + \cos(\pi - t)$, or $H(t) = 1 + t - \cos(t)$.

g. Susan and James decide to test Susan's model by actually rolling a wheel with radius 1 m. However, when the gum first touched the ground, it came off the wheel and stuck to the ground at that position. How horizontally far from the initial position is the gum? Verify that your function from part (f) predicts this answer, too.

The gum is on the ground at time $\frac{3\pi}{2}$ seconds, and the wheel has rolled $\frac{3\pi}{2}$ meters during this time. The center of the wheel is at x-coordinate $1 + \frac{3\pi}{2}$ meters, which is also the x-coordinate of the gum (which is directly below the center at this time). Using the formula $H(t) = 1 + t - \cos(t)$ from part (f) gives
$H\left(\frac{3\pi}{2}\right) = 1 + \frac{3\pi}{2} - \cos\left(\frac{3\pi}{2}\right) = 1 + \frac{3\pi}{2} + 0$. These agree.

4. Betty was looking at the Pythagorean Identity: for all real numbers θ,

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

a. Betty used the Pythagorean identity to make up the equation below. She then stated, "Wow, I've discovered a new identity that is true for all θ." Do you agree with her? Why or why not?

$$\frac{\sin^2(\theta)}{1 - \cos(\theta)} = 1 + \cos(\theta)$$

This is not an identity for all values of θ because the expressions on each side of the equation, thought of as functions, have different domains. The domain of the function on the left excludes 0 (and other places where the cosine is 1), whereas the one on the right does not.

b. Prove the Pythagorean identity.

For any real number θ, rotate the initial ray (given by the positive x-axis) by θ radians. The image of the initial ray under that rotation intersects the unit circle at the point (x, y). By definition, $\cos(\theta) = x$ and $\sin(\theta) = y$. However, the point (x, y) is also a point on the unit circle, so it satisfies the equation $x^2 + y^2 = 1$. Substituting $\cos(\theta)$ for x and $\sin(\theta)$ for y into this equation yields the desired identity.

c. The real number θ is such that $\sin(\theta) = 0.6$. Calculate $|\cos(\theta)|$ and $|\tan(\theta)|$.

$\sin^2(\theta) + \cos^2(\theta) = 1$
$0.36 + \cos^2(\theta) = 1$
$\cos^2(\theta) = 0.64$
$|\cos(\theta)| = 0.8$
$|\tan(\theta)| = \frac{0.6}{0.8} = 0.75.$

d. Suppose additional information is given about the number θ from part (c). You are told that $\frac{\pi}{2} < \theta < \pi$. What are the values of $\cos(\theta)$ and $\tan(\theta)$? Explain.

The cosine and tangent are negative for rotations that place the terminal ray in the second quadrant. So, the cosine and tangent are $\cos(\theta) = -0.8$ and $\tan(\theta) = -0.75$.