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## Congruence, Proof, and Constructions

### Module Overview

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Topics A through G (assessment 1 day, return 2 days, remediation or further applications 3 days)
Module Overview

Module 1 embodies critical changes in Geometry as outlined by the Common Core. The heart of the module is the study of transformations and the role transformations play in defining congruence.

Students begin this module with Topic A, Basic Constructions. Major constructions include an equilateral triangle, an angle bisector, and a perpendicular bisector. Students synthesize their knowledge of geometric terms with the use of new tools and simultaneously practice precise use of language and efficient communication when they write the steps that accompany each construction (G.CO.A.1).

Constructions segue into Topic B, Unknown Angles, which consists of unknown angle problems and proofs. These exercises consolidate students’ prior body of geometric facts and prime students’ reasoning abilities as they begin to justify each step for a solution to a problem. Students began the proof writing process in Grade 8 when they developed informal arguments to establish select geometric facts (8.G.A.5).

Topics C and D, Transformations/Rigid Motions and Congruence, build on students’ intuitive understanding developed in Grade 8. With the help of manipulatives, students observed how reflections, translations, and rotations behave individually and in sequence (8.G.A.1, 8.G.A.2). In high school Geometry, this experience is formalized by clear definitions (G.CO.A.4 and more in-depth exploration (G.CO.A.3, G.CO.A.5). The concrete establishment of rigid motions also allows proofs of facts formerly accepted to be true (G.CO.C.9). Similarly, students’ Grade 8 concept of congruence transitions from a hands-on understanding (8.G.A.2) to a precise, formally notated understanding of congruence (G.CO.B.6). With a solid understanding of how transformations form the basis of congruence, students next examine triangle congruence criteria. Part of this examination includes the use of rigid motions to prove how triangle congruence criteria such as SAS actually work (G.CO.B.7, G.CO.B.8).

In Topic E, Proving Properties of Geometric Figures, students use what they have learned in Topics A through D to prove properties—those that have been accepted as true and those that are new—of parallelograms and triangles (G.CO.C.10, G.CO.C.11). The module closes with a return to constructions in Topic F (G.CO.D.13), followed by a review of the module that highlights how geometric assumptions underpin the facts established thereafter (Topic G).

Focus Standards

Experiment with transformations in the plane.

G-CO.A.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Understand congruence in terms of rigid motions.

G-CO.B.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

G-CO.B.7 Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

G-CO.B.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems.

G-CO.C.9 Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment’s endpoints.

G-CO.C.10 Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

G-CO.C.11 Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.
Make geometric constructions.

G-CO.D.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

G-CO.D.13 Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Foundational Standards

Understand congruence and similarity using physical models, transparencies, or geometry software.

8.G.A.1 Verify experimentally the properties of rotations, reflections, and translations:
   a. Lines are taken to lines, and line segments to line segments of the same length.
   b. Angles are taken to angles of the same measure.
   c. Parallel lines are taken to parallel lines.

8.G.A.2 Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

8.G.A.3 Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.

8.G.A.5 Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*

Focus Standards for Mathematical Practice

MP.3 Construct viable arguments and critique the reasoning of others. Students articulate the steps needed to construct geometric figures, using relevant vocabulary. Students develop and justify conclusions about unknown angles and defend their arguments with geometric reasons.

MP.4 Model with mathematics. Students apply geometric constructions and knowledge of rigid motions to solve problems arising with issues of design or location of facilities.

MP.5 Use appropriate tools strategically. Students consider and select from a variety of tools in constructing geometric diagrams, including (but not limited to) technological tools.
MP.6 **Attend to precision.** Students precisely define the various rigid motions. Students demonstrate polygon congruence, parallel status, and perpendicular status via formal and informal proofs. In addition, students clearly and precisely articulate steps in proofs and constructions throughout the module.

**Terminology**

**New or Recently Introduced Terms**

- **Isometry** (An *isometry* of the plane is a transformation of the plane that is distance-preserving.)

**Familiar Terms and Symbols**

- Congruence
- Reflection
- Rotation
- Transformation
- Translation

**Suggested Tools and Representations**

- Compass and straightedge
- Geometer’s Sketchpad or Geogebra Software
- Patty paper

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2These are terms and symbols students have seen previously.
Preparing to Teach a Module

Preparation of lessons will be more effective and efficient if there has been an adequate analysis of the module first. Each module in *A Story of Functions* can be compared to a chapter in a book. How is the module moving the plot, the mathematics, forward? What new learning is taking place? How are the topics and objectives building on one another? The following is a suggested process for preparing to teach a module.

Step 1: Get a preview of the plot.

A: Read the Table of Contents. At a high level, what is the plot of the module? How does the story develop across the topics?

B: Preview the module’s Exit Tickets to see the trajectory of the module’s mathematics and the nature of the work students are expected to be able to do.

Note: When studying a PDF file, enter “Exit Ticket” into the search feature to navigate from one Exit Ticket to the next.

Step 2: Dig into the details.

A: Dig into a careful reading of the Module Overview. While reading the narrative, liberally reference the lessons and Topic Overviews to clarify the meaning of the text—the lessons demonstrate the strategies, show how to use the models, clarify vocabulary, and build understanding of concepts.

B: Having thoroughly investigated the Module Overview, read through the Student Outcomes of each lesson (in order) to further discern the plot of the module. How do the topics flow and tell a coherent story? How do the outcomes move students to new understandings?

Step 3: Summarize the story.

Complete the Mid- and End-of-Module Assessments. Use the strategies and models presented in the module to explain the thinking involved. Again, liberally reference the lessons to anticipate how students who are learning with the curriculum might respond.
Preparing to Teach a Lesson

A three-step process is suggested to prepare a lesson. It is understood that at times teachers may need to make adjustments (customizations) to lessons to fit the time constraints and unique needs of their students. The recommended planning process is outlined below. Note: The ladder of Step 2 is a metaphor for the teaching sequence. The sequence can be seen not only at the macro level in the role that this lesson plays in the overall story, but also at the lesson level, where each rung in the ladder represents the next step in understanding or the next skill needed to reach the objective. To reach the objective, or the top of the ladder, all students must be able to access the first rung and each successive rung.

Step 1: Discern the plot.
   A: Briefly review the module’s Table of Contents, recalling the overall story of the module and analyzing the role of this lesson in the module.
   B: Read the Topic Overview related to the lesson, and then review the Student Outcome(s) and Exit Ticket of each lesson in the topic.
   C: Review the assessment following the topic, keeping in mind that assessments can be found midway through the module and at the end of the module.

Step 2: Find the ladder.
   A: Work through the lesson, answering and completing each question, example, exercise, and challenge.
   B: Analyze and write notes on the new complexities or new concepts introduced with each question or problem posed; these notes on the sequence of new complexities and concepts are the rungs of the ladder.
   C: Anticipate where students might struggle, and write a note about the potential cause of the struggle.
   D: Answer the Closing questions, always anticipating how students will respond.

Step 3: Hone the lesson.
   Lessons may need to be customized if the class period is not long enough to do all of what is presented and/or if students lack prerequisite skills and understanding to move through the entire lesson in the time allotted. A suggestion for customizing the lesson is to first decide upon and designate each question, example, exercise, or challenge as either “Must Do” or “Could Do.”
   A: Select “Must Do” dialogue, questions, and problems that meet the Student Outcome(s) while still providing a coherent experience for students; reference the ladder. The expectation should be that the majority of the class will be able to complete the “Must Do” portions of the lesson within the allocated time. While choosing the “Must Do” portions of the lesson, keep in mind the need for a balance of dialogue and conceptual questioning, application problems, and abstract problems, and a balance between students using pictorial/graphical representations and abstract representations. Highlight dialogue to be included in the delivery of instruction so that students have a chance to articulate and consolidate understanding as they move through the lesson.
B: “Must Do” portions might also include remedial work as necessary for the whole class, a small group, or individual students. Depending on the anticipated difficulties, the remedial work might take on different forms as suggested in the chart below.

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<th>“Must Do” Remedial Problem Suggestion</th>
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<td>The first problem of the lesson is too challenging.</td>
<td>Write a short sequence of problems on the board that provides a ladder to Problem 1. Direct students to complete those first problems to empower them to begin the lesson.</td>
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<tr>
<td>There is too big of a jump in complexity between two problems.</td>
<td>Provide a problem or set of problems that bridge student understanding from one problem to the next.</td>
</tr>
<tr>
<td>Students lack fluency or foundational skills necessary for the lesson.</td>
<td>Before beginning the lesson, do a quick, engaging fluency exercise³. Before beginning any fluency activity for the first time, assess that students have conceptual understanding of the problems in the set and that they are poised for success with the easiest problem in the set.</td>
</tr>
<tr>
<td>More work is needed at the concrete or pictorial level.</td>
<td>Provide manipulatives or the opportunity to draw solution strategies.</td>
</tr>
<tr>
<td>More work is needed at the abstract level.</td>
<td>Add a set of abstract problems to be completed toward the end of the lesson.</td>
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C: “Could Do” problems are for students who work with greater fluency and understanding and can, therefore, complete more work within a given time frame.

D: At times, a particularly complex problem might be designated as a “Challenge!” problem to provide to advanced students. Consider creating the opportunity for students to share their “Challenge!” solutions with the class at a weekly session or on video.

E: If the lesson is customized, be sure to carefully select Closing questions that reflect such decisions, and adjust the Exit Ticket if necessary.

### Assessment Summary

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