Lesson 26: Solving Rational Equations

Student Outcomes

- Students solve rational equations, monitoring for the creation of extraneous solutions.

Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions so that in this lesson they can solve equations involving rational expressions (**A-REI.A.2**). The skills developed in this lesson are required to solve equations of the form $f(x) = c$ for a rational function f and constant c in Lesson 27 and later in Module 3 (**F-BF.B.4a**).

There is more than one approach to solving a rational equation, and this section explores two such methods. The first method is to multiply both sides by the common denominator to clear fractions. The second method is to find equivalent forms of all expressions with a common denominator, set the numerators equal to each other, and solve the resulting equation. Either approach requires keeping an eye out for extraneous solutions; in other words, values that appear to be a solution to the equation but cause division by zero and are, thus, not valid. Throughout the work with rational expressions, students analyze the structure of the expressions in order to decide on their next algebraic steps (MP.7). Encourage students to check their answers by substituting their solutions back into each side of the equation separately.

Classwork

Exercises 1–2 (8 minutes)

Let students solve this any way they can, and then discuss their answers. Focus on adding the fractions on the left and equating numerators or multiplying both sides by a common multiple. Indicate a practical technique of finding a common denominator. These first two exercises highlight MP.7, as students must recognize the given expressions to be of the form $\frac{a}{b} + \frac{c}{b}$ or $\frac{a}{b} + \frac{c}{d}$; by expressing the equations in the simplified form $\frac{A}{B} = \frac{C}{B}$, they realize that we must have $A = C$.

> **Scaffolding:**
>
> Struggling students may benefit from first solving the equation $\frac{x}{5} - \frac{2}{5} = \frac{1}{5}$.
>
> More advanced students may try to solve
>
> $$\frac{x-1}{x+2} = \frac{3}{4}.$$

Exercises 1–2

Solve the following equations for x, and give evidence that your solutions are correct.

1. $\dfrac{x}{2} + \dfrac{1}{3} = \dfrac{5}{6}$

   Combining the expressions on the left, we have $\dfrac{3x}{6} + \dfrac{2}{6} = \dfrac{5}{6}$, so $\dfrac{3x+2}{6} = \dfrac{5}{6}$; therefore, $3x + 2 = 5$. Then, $x = 1$.

   Or, using another approach: $6 \cdot \left(\dfrac{x}{2} + \dfrac{1}{3}\right) = 6 \cdot \left(\dfrac{5}{6}\right)$, so $3x + 2 = 5$; then, $x = 1$.

   The solution to this equation is 1. To verify, we see that when $x = 1$, we have $\dfrac{x}{2} + \dfrac{1}{3} = \dfrac{1}{2} + \dfrac{1}{3} = \dfrac{3}{6} + \dfrac{2}{6} = \dfrac{5}{6}$, so 1 is a valid solution.

---

2. $\dfrac{2x}{9} + \dfrac{5}{9} = \dfrac{8}{9}$

Since the expressions already have a common denominator, we see that $\dfrac{2x}{9} + \dfrac{5}{9} = \dfrac{2x+5}{9}$, so we need to solve

$\dfrac{2x+5}{9} = \dfrac{8}{9}$. It then follows that the numerators are equal, so $2x + 5 = 8$. Solving for x gives $x = \dfrac{3}{2}$. To verify, we

see that when $x = \dfrac{3}{2}$, we have $\dfrac{2x}{9} + \dfrac{5}{9} = \dfrac{2\left(\frac{3}{2}\right)}{9} + \dfrac{5}{9} = \dfrac{3}{9} + \dfrac{5}{9} = \dfrac{8}{9}$; thus, $\dfrac{3}{2}$ is a valid solution.

Remind students that two rational expressions with the same denominator are equal if the numerators are equal.

Discussion (2 minutes)

Now that students know how to add, subtract, multiply, and divide rational expressions, it is time to use some of those basic operations to solve equations involving rational expressions. An equation involving rational expressions is called a *rational equation*. Keeping the previous exercise in mind, this section looks at two different approaches to solving rational equations.

Example (6 minutes)

MP.1

Ask students to try to solve this problem on their own. Have them discuss and explain their methods in groups or with neighbors. The teacher should circulate and lead a discussion of both methods once students have had a chance to try solving on their own.

Example

Solve the following equation: $\dfrac{x+3}{12} = \dfrac{5}{6}$.

Equating Numerators Method: Obtain expressions on both sides with the same denominator and equate numerators.

$$\frac{x+3}{12} = \frac{5}{6} \cdot \frac{2}{2}$$

$$\frac{x+3}{12} = \frac{10}{12}$$

Thus, $x + 3 = 10$, and $x = 7$; therefore, 7 is the solution to our original equation.

Clearing Fractions Method: Multiply both sides by a common multiple of the denominators to clear the fractions, and then solve the resulting equation.

$$12 \cdot \left(\frac{x+3}{12}\right) = 12 \cdot \left(\frac{5}{6}\right)$$

$$x + 3 = 10$$

We can see, once again, that the solution is 7.

EUREKA
MATH™

Discussion (3 minutes)

Ask students to discuss both methods used in the previous example. Which method do they prefer, and why? Does one method seem to be more efficient than the other? Have a few groups report their opinions to the class. At no time should students be required to use a particular method; just be sure they understand both approaches, and allow them to use whichever method seems more natural.

Exercise 3 (6 minutes)

Remind students that when saying a *solution* to an equation, this refers to a value of the variable, usually x, which results in a true number sentence. In Lesson 22, students learned that there are some values of the variable that are not allowed in order to avoid division by zero. Before students start working on the following exercise, ask them to identify the values of x that must be excluded. Wait for students to respond that $x \neq 0$ and $x \neq 2$ before having them work with a partner on the following exercise.

MP.7

Exercises 3–7

3. Solve the following equation: $\dfrac{3}{x} = \dfrac{8}{x-2}$.

Method 1: Convert both expressions to equivalent expressions with a common denominator. The common denominator is $x(x-2)$, so we use the identity property of multiplication to multiply the left side by $\dfrac{x-2}{x-2}$ and the right side by $\dfrac{x}{x}$. This does not change the value of the expression on either side of the equation.

$$\left(\frac{x-2}{x-2}\right) \cdot \left(\frac{3}{x}\right) = \left(\frac{x}{x}\right) \cdot \left(\frac{8}{x-2}\right)$$

$$\frac{3x-6}{x(x-2)} = \frac{8x}{x(x-2)}$$

Since the denominators are equal, we can see that the numerators must be equal; thus, $3x - 6 = 8x$. Solving for x gives a solution of $-\dfrac{6}{5}$. At the outset of this example, we noted that x cannot take on the value of 0 or 2, but there is nothing preventing x from taking on the value $-\dfrac{6}{5}$. Thus, we have found a solution. We can check our work. Substituting $-\dfrac{6}{5}$ into $\dfrac{3}{x}$ gives us $\dfrac{3}{\left(-6/5\right)} = -\dfrac{5}{2}$, and substituting $-\dfrac{6}{5}$ into $\dfrac{8}{x-2}$ gives us $\dfrac{8}{\left(-6/5\right)-2} = -\dfrac{5}{2}$. Thus, when $x = -\dfrac{6}{5}$, we have $\dfrac{3}{x} = \dfrac{8}{x-2}$; therefore, $-\dfrac{6}{5}$ is indeed a solution.

Method 2: Multiply both sides of the equation by the common denominator $x(x-2)$, and solve the resulting equation.

$$x(x-2)\left(\frac{3}{x}\right) = x(x-2)\left(\frac{8}{x-2}\right)$$

$$3(x-2) = 8x$$

$$3x - 6 = 8x$$

From this point, we follow the same steps as we did in Method 1, and we get the same solution: $-\dfrac{6}{5}$.

Exercise 4 (6 minutes)

Have students continue to work with partners to solve the following equation. Walk around the room and observe student progress; if necessary, offer the following hints and reminders:

- Reminder: Ask students to identify excluded values of a. Suggest that they factor the denominator $a^2 - 4$. They should discover that $a \neq 2$ and $a \neq -2$ must be specified.

- Hint 1: Ask students to identify a common denominator of the three expressions in the equation. They should respond with $(a - 2)(a + 2)$, or equivalently, $a^2 - 4$.

- Hint 2: What do we need to do with this common denominator? They should determine that they need to find equivalent rational expressions for each of the terms with denominator $(a - 2)(a + 2)$.

---

4.   Solve the following equation for a: $\dfrac{1}{a+2} + \dfrac{1}{a-2} = \dfrac{4}{a^2-4}$.

First, we notice that we must have $a \neq 2$ and $a \neq -2$. Then, we apply Method 1:

$$\left(\frac{a-2}{a-2}\right) \cdot \left(\frac{1}{a+2}\right) + \left(\frac{a+2}{a+2}\right) \cdot \left(\frac{1}{a-2}\right) = \frac{4}{(a-2)(a+2)}$$

$$\frac{a-2}{(a-2)(a+2)} + \frac{a+2}{(a-2)(a+2)} = \frac{4}{(a-2)(a+2)}$$

$$\frac{2a}{(a-2)(a+2)} = \frac{4}{(a-2)(a+2)}.$$

Since the denominators are equal, we know that the numerators are equal; thus, we have $2a = 4$, which means that $a = 2$. Thus, the only solution to this equation is 2. However, a is not allowed to be 2 because if $a = 2$, then $\dfrac{1}{a-2}$ is not defined. This means that the original equation, $\dfrac{1}{a+2} + \dfrac{1}{a-2} = \dfrac{4}{a^2-4}$, has no solution.

Scaffolding:

Let students know that the word *extraneous* has meaning outside of the mathematics classroom; ask them to guess its definition, and then provide the actual definition.

EXTRANEOUS: Irrelevant or unrelated to the subject being dealt with.

Students may benefit from choral repetition, as well as a visual representation of this word.

Introduce the term *extraneous solution*. An invalid solution that may arise when we manipulate a rational expression is called an extraneous solution. An extraneous solution is a value that satisfies a transformed equation but does not satisfy the original equation.

Exercises 5–7 (8 minutes)

Give students a few minutes to discuss extraneous solutions with a partner. When do they occur, and how do you know when you have one? Extraneous solutions occur when one of the solutions found does not make a true number sentence when substituted into the original equation. The only way to know there is one is to note the values of the variable that will cause a part of the equation to be undefined. This lesson is concerned with division by zero; later lessons exclude values of the variable that would cause the square root of a negative number. Make sure that all students have an understanding of extraneous solutions before proceeding. Then, have them work in pairs on the following exercises.

5. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{4}{3x} + \frac{5}{4} = \frac{3}{x}$$

First, note that we must have $x \neq 0$.

Equating numerators: $\dfrac{16}{12x} + \dfrac{15x}{12x} = \dfrac{36}{12x}$ *Then, we have $16 + 15x = 36$, and the solution is $x = \dfrac{4}{3}$.*

Clearing fractions: $12x\left(\dfrac{4}{3x} + \dfrac{5}{4}\right) = 12x\left(\dfrac{3}{x}\right)$ *Then, we have $16 + 15x = 36$, and the solution is $x = \dfrac{4}{3}$.*

The solution $\dfrac{4}{3}$ is valid since the only excluded value is 0.

6. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b-2}{b^2-9}$$

First, note that we must have $x \neq 3$ and $x \neq -3$.

Equating numerators: $\dfrac{7(b-3)}{(b-3)(b+3)} + \dfrac{5(b+3)}{(b-3)(b+3)} = \dfrac{10b-2}{(b-3)(b+3)}$

Matching numerators, we have $7b - 21 + 5b + 15 = 10b - 2$, which leads to $2b = 4$; therefore, $b = 2$.

Clearing fractions: $(b-3)(b+3)\left(\dfrac{7}{b+3} + \dfrac{5}{b-3}\right) = (b-3)(b+3)\left(\dfrac{10b-2}{b^2-9}\right)$

We have $7(b-3) + 5(b+3) = 10b - 2$, which leads to $2b = 4$; therefore, $b = 2$.

The solution 2 is valid since the only excluded values are 3 and -3.

7. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{1}{x-6} + \frac{x}{x-2} = \frac{4}{x^2 - 8x + 12}$$

First, note that we must have $x \neq 6$ and $x \neq 2$.

Equating numerators:

$$\frac{x-2}{(x-6)(x-2)} + \frac{x^2 - 6x}{(x-6)(x-2)} = \frac{4}{(x-6)(x-2)}$$
$$x^2 - 5x - 2 = 4$$
$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$

The solutions are 6 and -1.

Clearing fractions:

$$\left(\frac{1}{x-6} + \frac{x}{x-2}\right)(x-6)(x-2) = \left(\frac{4}{(x-6)(x-2)}\right)(x-6)(x-2)$$
$$(x-2) + x(x-6) = 4$$
$$x^2 - 6x + x - 2 = 4$$
$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$

The solutions are 6 and -1.

Because x is not allowed to be 6 in order to avoid division by zero, the solution 6 is extraneous; thus, -1 is the only solution to the given rational equation.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to explain how to identify extraneous solutions and why they arise when solving rational equations.

Lesson Summary

In this lesson, we applied what we have learned in the past two lessons about addition, subtraction, multiplication, and division of rational expressions to solve rational equations. An extraneous solution is a solution to a transformed equation that is not a solution to the original equation. For rational functions, extraneous solutions come from the excluded values of the variable.

Rational equations can be solved one of two ways:

1. **Write each side of the equation as an equivalent rational expression with the same denominator and equate the numerators. Solve the resulting polynomial equation, and check for extraneous solutions.**

2. **Multiply both sides of the equation by an expression that is the common denominator of all terms in the equation. Solve the resulting polynomial equation, and check for extraneous solutions.**

Exit Ticket (4 minutes)

Name _____   Date _____

Lesson 26: Solving Rational Equations

Exit Ticket

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2-9}$$

Exit Ticket Sample Solutions

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2-9}$$

First, note that we must have $x \neq 3$ and $x \neq -3$.

Using the equating numerators method: $\frac{7(b-3)}{(b-3)(b+3)} + \frac{5(b+3)}{(b-3)(b+3)} = \frac{10b}{(b-3)(b+3)}$

Matching numerators, we have $7b - 21 + 5b + 15 = 10b$, which leads to $12b - 6 = 10b$; therefore, $b = 3$.

However, since the excluded values are 3 and -3, the solution 3 is an extraneous solution, and there is no solution to
$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2-9}$.

Problem Set Sample Solutions

1. **Solve the following equations, and check for extraneous solutions.**

   a. $\frac{x-8}{x-4} = 2$

      0

   b. $\frac{4x-8}{x-2} = 4$

      All real numbers except 2

   c. $\frac{x-4}{x-3} = 1$

      No solution

   d. $\frac{4x-8}{x-2} = 3$

      No solution

   e. $\frac{1}{2a} - \frac{2}{2a-3} = 0$

      $-\frac{3}{2}$

   f. $\frac{3}{2x+1} = \frac{5}{4x+3}$

      -2

   g. $\frac{4}{x-5} - \frac{2}{5+x} = \frac{2}{x}$

      $-\frac{5}{3}$

   h. $\frac{y+2}{3y-2} + \frac{y}{y-1} = \frac{2}{3}$

      $\frac{5}{6}, -2$

   i. $\frac{3}{x+1} - \frac{2}{1-x} = 1$

      $0, 5$

   j. $\frac{4}{x-1} + \frac{3}{x} - 3 = 0$

      $\frac{1}{3}, 3$

   k. $\frac{x+1}{x+3} - \frac{x-5}{x+2} = \frac{17}{6}$

      $0, -\frac{55}{17}$

   l. $\frac{x+7}{4} - \frac{x+1}{2} = \frac{5-x}{3x-14}$

      $5, 6$

   m. $\frac{b^2-b-6}{b^2} - \frac{2b+12}{b} = \frac{b-39}{2b}$

      $3, \frac{4}{3}$

   n. $\frac{1}{p(p-4)} + 1 = \frac{p-6}{p}$

      $\frac{23}{6}$

   o. $\frac{1}{h+3} = \frac{h+4}{h-2} + \frac{6}{h-2}$

      $-8, -4$

   p. $\frac{m+5}{m^2+m} = \frac{1}{m^2+m} - \frac{m-6}{m+1}$

      $4, 1$

2. Create and solve a rational equation that has 0 as an extraneous solution.

One such equation is $\dfrac{1}{x-1} + \dfrac{1}{x} = \dfrac{1}{x-x^2}$.

3. Create and solve a rational equation that has 2 as an extraneous solution.

One such equation is $\dfrac{1}{x-2} + \dfrac{1}{x+2} = \dfrac{4}{x^2-4}$.

Extension:

4. Two lengths a and b, where $a > b$, are in *golden ratio* if the ratio of $a + b$ is to a is the same as a is to b. Symbolically, this is expressed as $\dfrac{a}{b} = \dfrac{a+b}{a}$. We denote this common ratio by the Greek letter *phi (pronounced "fee")* with symbol φ, so that if a and b are in common ratio, then $\varphi = \dfrac{a}{b} = \dfrac{a+b}{a}$. By setting $b = 1$, we find that $\varphi = a$ and φ is the positive number that satisfies the equation $\varphi = \dfrac{\varphi+1}{\varphi}$. Solve this equation to find the numerical value for φ.

a+b is to a as a is to b
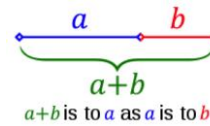
We can apply either method from the previous lesson to solve this equation.

$$\varphi = \frac{\varphi + 1}{\varphi}$$

$$\varphi^2 = \varphi + 1$$

$$\varphi^2 - \varphi - 1 = 0$$

Applying the quadratic formula, we have two solutions:

$$\varphi = \frac{1+\sqrt{5}}{2} \ \ or \ \ \varphi = \frac{1-\sqrt{5}}{2}.$$

Since φ is a positive number, and $\dfrac{1-\sqrt{5}}{2} < 0$, we have $\varphi = \dfrac{1+\sqrt{5}}{2}$.

5. Remember that if we use x to represent an integer, then the next integer can be represented by $x + 1$.

   a. Does there exist a pair of consecutive integers whose reciprocals sum to $\dfrac{5}{6}$? Explain how you know.

   Yes, 2 and 3 because $\dfrac{1}{2} + \dfrac{1}{3} = \dfrac{3}{6} + \dfrac{2}{6} = \dfrac{5}{6}$.

b. **Does there exist a pair of consecutive integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know.**

If x represents the first integer, then $x + 1$ represents the next integer. Suppose $\frac{1}{x} + \frac{1}{x+1} = \frac{3}{4}$. Then,

$$\frac{1}{x} + \frac{1}{x+1} = \frac{3}{4}$$

$$\frac{4(x+1) + 4x}{4x(x+1)} = \frac{3x(x+1)}{4x(x+1)}$$

$$8x + 4 = 3x^2 + 3x$$

$$3x^2 - 5x - 4 = 0.$$

The solutions to this quadratic equation are $\frac{5+\sqrt{73}}{6}$ and $\frac{5-\sqrt{73}}{6}$, so there are no integers that solve this equation. Thus, there are no pairs of consecutive integers whose reciprocals sum to $\frac{3}{4}$.

c. **Does there exist a pair of consecutive *even* integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know.**

If x represents the first integer, then $x + 2$ represents the next even integer. Suppose $\frac{1}{x} + \frac{1}{x+2} = \frac{3}{4}$. Then,

$$\frac{1}{x} + \frac{1}{x+2} = \frac{3}{4}$$

$$\frac{4(x+2) + 4x}{4x(x+2)} = \frac{3x(x+2)}{4x(x+2)}$$

$$8x + 8 = 3x^2 + 6x$$

$$3x^2 - 2x - 8 = 0.$$

The solutions to this quadratic equation are $-\frac{4}{3}$ and 2; therefore, the only even integer x that solves the equation is 2. Then, 2 and 4 are consecutive even integers whose reciprocals sum to $\frac{3}{4}$.

d. **Does there exist a pair of consecutive *even* integers whose reciprocals sum to $\frac{5}{6}$? Explain how you know.**

If x represents the first integer, then $x + 2$ represents the next even integer. Suppose $\frac{1}{x} + \frac{1}{x+2} = \frac{5}{6}$. Then,

$$\frac{1}{x} + \frac{1}{x+2} = \frac{5}{6}$$

$$\frac{6(x+2) + 6x}{6x(x+2)} = \frac{5x(x+2)}{6x(x+2)}$$

$$12x + 12 = 5x^2 + 10x$$

$$5x^2 - 2x - 12 = 0.$$

The solutions to this quadratic equation are $\frac{1+\sqrt{61}}{5}$ and $\frac{1-\sqrt{61}}{5}$, so there are no integers that solve this equation. Thus, there are no pairs of consecutive even integers whose reciprocals sum to $\frac{5}{6}$.

EUREKA
MATH™